Global Analytic Approach to Super Teichmüller Spaces

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Christoph Sachse

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Die Annahme der Dissertation haben empfohlen:

1. Prof. Dr. Jürgen Jost (MPIMiS, Leipzig)
2. Prof. Dr. Dimitri Leites (Stockholm University, Schweden)
3. Prof. Dr. Dan Freed (University of Texas at Austin, USA)

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Chapter 1

Introduction

Falls ein Wechsel
des Universums erforderlich ist,
werden wir das anzeigen.
H. Schubert, Kategorien I

In this thesis, we adopt and investigate a new formalism for supergeometry which focuses on the categorical and homological properties of the theory, and apply it to the construction of Teichmüller spaces of certain superconformal structures. Supergeometry in its mathematically rigorous form is usually formulated as a theory of locally ringed topological spaces, where all rings involved are supercommutative. Once one has established the definition of a supermanifold as a ringed space (which, as algebraic geometry tells us, is basically inevitable), one has the full arsenal of tools from algebraic geometry at hand. The availability of many of these tools is not just a nice side-effect, many of them are actually indispensible to making the whole theory viable. One such tool is the functor of points, which is a well-known method of homological algebra allowing one to define an object of a category via the morphisms of all other objects of this category to it. The functor of points is, e.g., needed to define what a Lie supergroup is. Its usefulness had been realized already at a very early stage, at least among those supergeometers who used the ringed space approach, such as Leites [Lei80], [Leiar], Bernstein [DM99], Manin [Man97], [Man91] and others. Molotkov [Mol84] was the first to publish a full-blown reformulation of supergeometry in categorical terms, the main purpose of which was to enable the construction of infinite-dimensional superspaces. Unfortunately, his amazing preprint [Mol84] contains no proofs, and its dense and abstract style made it difficult to approach at least for many of the more physically oriented researchers in the field. Therefore, no applications of his approach have been published so far, and infinite-dimensional supergeometry was very rarely used, although physics offers plenty of opportunities for applications.

1.1 Supersymmetry and supergeometry

Superalgebra and supergeometry were invented in the early seventies as tools for the path integral formalism of quantum field theories involving fermion fields. The pioneer of this subject was undoubtedly Berezin, who, being concerned with the method of second quantization, already realized in the sixties that one could use algebras with anticommuting generators to unify the description of boson and fermion fields. Algebras containing anticommuting along with commuting generators were, of course, known long ago. The archetype of such an algebra is the Grassmann algebra Λ_n, which is generated by n "odd", i.e., anticommuting elements $\theta_1, \dots, \theta_n$. It is free except for the relations

$$\theta_i \theta_j = -\theta_j \theta_i$$

which imply $\theta_i^2 = 0$. This algebra can also be viewed simply as the exterior algebra of a vector space V of dimension n. So in the realm of linear and commutative algebra, the step from ordinary algebras to superalgebras is relatively easy: a superalgebra is defined as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra whose operations and morphisms preserve the grading. But it was also Berezin who first realized that it could be possible to extend the use of anticommuting quantities to analysis and geometry, which is far less obvious. The search for a concept of "supermanifold" indeed took several years, and was settled in 1972 by Leites [Lei80], [BL75]. The emergence of this new concept went disregarded for about two years, until Wess and Zumino [WZ74] presented their famous first supersymmetric field theories, which triggered an avalanche of research on supergeometry and its applications in physics. The majority of applications for supersymmetry still originates from high engery physics. Supersymmetric field theories have become standard tools during the past decades. The most famous applications are probably superstring theory and supergravity. But there are also a variety of applications in solid-state physics, see, e.g., the book by Efetov [Efe97].

Although supergeometry as it was invented in the seventies is formulated in a way which mimicks ordinary differential geometry as closely as possible, it can only really be understood with the help of algebraic geometry. An indispensible tool is the language of ringed spaces. Unlike an ordinary manifold, a supermanifold is not defined as a space which is locally homeomorphic to some linear space, although it is still locally isomorphic to a certain model space. The local rings of functions are not commutative algebras anymore, but rather supercommuative ones, which implies, in particular, that they are not reduced. This entails that a supermanifold is not fully described by its topological points, which one can recover as the set of maximal ideals of the structure sheaf. The notion of a "point" itself has to be handled with great care in this context. Nonetheless, concepts like coordinates, functions, vector fields, differential forms and integration can all be carried over to supergeometry, but their intuitive interpretation becomes more subtle. In particular the existence of some notion of coordinate makes it

possible to formally wrap up much of the theory in the same language as ordinary differential geometry.

This is an advantage for many applications and calculations, but the fact that the odd "dimensions" of supergeometry are really only algebraic, and not topological, creates some pitfalls. To hide the essentially algebraic nature of the odd variables behind a seemingly analytic formalism can sometimes obstruct a clear view on the problems. These subtleties, together with the generally higher level of abstraction enforced by the ringed space language, led to attempts to formulate supergeometry in different, less algebraic terms. An example is the de Witt approach to supermanifolds [DeW92] which tries to avoid the use of ringed spaces by formally reformulating ordinary geometry over the "supernumbers", which are assumed to be elements of a Grassmann algebra. In order to escape the "finite size effects" one can create by an unfortunate choice of this algebra, one often assumes it to be infinite. This, however, produces different types of problems, related to, e.g., convergence questions. As was already argued many times by the more algebraically oriented supergeometers, one must require *functoriality* with respect to an exchange of the algebra over which one works. If one adheres to this point of view, the de Witt approach becomes equivalent to the ringed space approach, and the de Witt topology on linear superspaces becomes the Grothendieck topology on the category of superdomains as proposed by Molotkov [Mol84].

In much of the physics literature, the rigorous ringed space approach has still not become standard, which has several reasons. One main reason is certainly the considerable technical difficulty that inevitably accompanies this approach. On the other hand, the supergeometric problems appearing in physics often do not require this whole massive machinery. Many problems can be treated locally, and then it is enough to manipulate a set of even and odd coordinates tailor-made for the situation at hand. Besides, odd quantities in physics are almost exclusively associated with fermion fields, i.e., they form spin representations of the underlying spacetime symmetry group (usually the Poincaré group). This is a very special type of supergeometry. In other words, supersymmetry and supergeometry are two rather different things. Supergeometry is a genuine extension of commutative geometry, achieved by using supercommutative rings and algebras instead of commutative ones. Supersymmetry is a certain feature of quantum field theories where superalgebra — or, in the case of supergravity, supergeometry — is only used to express this feature. Most applications of supergeometry, however, still originate in physics and are therefore often encoded in the language of quantum field theory, which to non-specialists is often cryptic and hard to decipher. It is maybe for this reason that, despite spectacular successes like [Wit82] and contributions by renowned mathematicians like Manin [Man97], [Man91] or Deligne [Del] and Bernstein [DM99], the mathematical community concerned with supergeometry has remained small. We hope that this work might also help in diminishing the prejudice that supergeometry is just a bizarre game with spinors invented by physicists, and show that it is a real extension of ordinary commu-

tative geometry which should indeed be of great interest to mathematicians, in particular differential and algebraic geometers.

1.2 The categorical approach

In this work we will strictly stick to the ringed space formalism. Besides this, we will work out the categorical approach to supermathematics. This is not a substitute for the explicit description in terms of sheaves, but rather an extension which allows us to study questions which would not be directly accessible in the standard formalism. Its investigation and the demonstration of its use for concrete problems is one of the main results of this thesis. Most of the methods of the categorical approach are actually well known in algebraic geometry and are in use, e.g., in the theory of schemes and non-reduced varieties [FGI+05], [Gro62]. As remarked above, this is an approach based not on the investigation of single objects but rather on their interplay with other objects of the category. It requires considerable technical machinery to establish. Nonetheless, we think that the gains in both conceptual insight and also completely new technical tools, e.g., infinite-dimensional supermanifolds, more than justify the work.

On the conceptual side, the categorical approach allows for making many of the cumbersome peculiarities of the ringed space formalism more transparent. For example, it allows us to enlighten the somewhat awkward role of "functions" on a supermanifold. They are, in analogy to classical geometry, sections of the structure sheaf, but have no interpretation as maps to the ground field. One often treats these sections as being "Grassmann-valued", i.e., to be actual maps into some Grassmann algebra Λ_n. But then, to avoid side-effects from the choice of the algebra, one has to adopt mysterious prescriptions like the one that the algebra must be "big enough" (usually meaning that it must contain at least one more generator than the number of odd quantities appearing in calculations). The categorical approach makes such unnatural conditions superfluous by simply requiring functoriality under an exchange of the algbra over which one works. The price to pay is that one then has to think of functions as functors taking values in *any* Grassmann algebra. That is, to think of them as being valued in some particular algebra is legitimate as long as one bears in mind that this means that one only handles one of their infinitely many components this way. For many practical problems this may, of course, be good enough, but it immediately becomes dangerous if one wants to define maps or morphisms by the values they take.

In the same vein, the functor of points clarifies the origin and necessity of the so-called odd parameters appearing when one works with supermanifolds. These parameters have been used in physics since the very first days of supersymmetry, but unfortunately most often without wondering where they actually come from. They usually take the form of "odd functions depending only on even variables". As an example, the transition function between a smooth superdomain with co-

ordinates (x, θ) and a domain with coordinates (y, η) would usually be written as

$$\begin{aligned} y &= f(x) + \psi(x)\theta \\ \eta &= \xi(x) + g(x)\theta, \end{aligned}$$

where f, g are ordinary smooth functions, but ψ, ξ are odd, i.e., anticommuting with other odd objects and nilpotent. But how can this happen for a function depending only on even quantities? And besides, how can they fit into the picture that this is the transition function of a superringed space, i.e., stalkwise a homomorphism of superalgebras? Again, one usually argues with Grassmann-valuedness here. And again, this is only almost true. To cleanly interpret the odd parameters, one has to think of a family of supermanifolds over a base supermanifold, rather than a single one. The odd coordinates of the base then become odd constants of the fibers of the family, providing the mysterious parameters. For brevity or other reasons, one usually does not denote them explicitly, but rather sticks to the rule that in all morphisms of superdomains, like the above, as many "odd functions" have to be introduced as necessary in order to not lose generality. If the base of the family is a superpoint, i.e. a supermanifold of the form $(\{*\}, \Lambda_n)$ where $\{*\}$ is the one-point space and Λ_n its sheaf of functions, then the structure sheaf of the family is just $\Lambda_n \otimes \mathcal{O}_{\mathcal{M}}$, where $\mathcal{M}$ is the fiber with sheaf $\mathcal{O}_{\mathcal{M}}$. This sheaf can indeed be interpreted as maps from $\mathcal{O}_{\mathcal{M}}$ to Λ_n, as one often does. But again, it is functoriality under exchange of the base which is the salient feature of this direct interpretation.

These two examples should make it clear that the categorical approach is not just a clever reformulation of the same old stuff, but really allows for a deeper and, in particular, cleaner understanding of supergeometry. Conceptual insight, however, is not the only advantage of this formalism. It also provides a whole variety of new technical tools and possibilities. Our main motivation to study it actually arose from the attempt to extend supergeometry to spaces of infinte dimensions. The ringed space approach is not directly applicable here. This is not a problem of supergeometry, but a general one, occurring, for example, also in complex geometry [Dou66]. The solution in the case of complex geometry is the use of so-called functored spaces, which are essentially an application of the functor of points to this particular problem. The defining property of an infinite-dimensional manifold is still that of being locally isomorphic to a linear space. But while in finite dimensions, there is — up to isomorphy — only one complex linear space to which the manifold can be locally identified, this is not true in the infinite-dimensional, e.g., Banach, context. One can still choose explicit coordinate maps to define the manifold structure. But to endow the underlying topological space with a sheaf of maps into $\mathbb{C}$ is not sufficient anymore. If one wants to stay in the ringed space picture, one now has to endow it with a functor from the category of complex Banach domains to the category of sheaves of holomorphic maps over the topological space in question.

To translate this to supergeometry, one first has to define linear supermanifolds of infinite dimension. This is where the functor of points comes in. The crucial difference from the ordinary complex case is that a supermanifold is not defined by its topological points, so bijections with open sets in super vector spaces will not solve the problem. If one switches to the functors of points, however, this intuition works again: two objects are isomorphic if and only if all of their point sets are isomorphic. We therefore define a supermanifold (of possibly infinite dimension) as a certain type of functor into the category of sets which is locally isomorphic to the functor of points of a linear supermanifold. Of course, one then has to specify what "locally" means for a functor. This is accomplished, following [Mol84], by introducing a Grothendieck pretopology on the appropriate functor category.

Another achievement of the categorical approach is that it allows a complete description of the diffeomorphism supergroup of a supermanifold. Already in the finite-dimensional cases supergroups are defined as group objects in the category of supermanifolds, i.e., in categorical terms. Formally any supergroup is therefore defined by a property of its functor of points, but in most cases one can construct an explicit ringed space and afterwards show that it represents this functor. In the infinite-dimensional case this is not possible anymore — here the functor of points must indeed be put to use.

1.3 Super Riemann surfaces

Our initial motivation for studying infinite-dimensional supergeometry was the Teichmüller theory of so-called $N = 1$ super Riemann surfaces. They are a particular species of complex $1|1$-dimensional supermanifolds appearing as world-sheets in $N = 1$ superstring theory. In fact, there are several types of $1|1$-dimensional supermanifolds which could legitimately be called analogues of Riemann surfaces, since in the super context, there are several types of superconformal structures. The one appearing in 2D supergravity and superstring theory is related to the superconformal algebra $\mathfrak{k}^L(1|1)$, a superversion of the agebra of contact vector fields.

Moduli problems can be formulated in supergeometry in precisely the same way as in ordinary geometry, namely as the search for universal parameter spaces for families of the objects in question. These moduli spaces can then, of course, also be supermanifolds, i.e., a superobject may have even as well as odd deformations. Since the underlying surface of a complex supersurface is a Riemann surface, one can then also study Teichmüller spaces by fixing a marking on the underlying surface. It was known from works of Vaintrob [Vai88a], LeBrun and Rothstein [LR88] and Crane and Rabin [CR88] that $\mathfrak{k}^L(1|1)$-surfaces indeed possess a Teichmüller space which has dimension $3g-3|2g-2$. It was, however, shown that one either has to construct this space as a quotient of a supermanifold by a certain $\mathbb{Z}_2$-action, i.e., a "superorbifold" [LR88], or that one has to settle with a supermanifold that only paratrizes a semiuniversal family. These results were

obtained mainly with the help of Kodaira-Spencer deformation theory, adapted to the context of supergeometry. A problem with this approach is that it gives only local information about the moduli spaces, since deformation theory only constructs infinitesimal neighbourhoods of a given structure.

The idea of this thesis was to attempt to carry a global approach to the construction of the Teichmüller space of Riemann surfaces invented by Fischer and Tromba [Tro92] over to supergeometry. Their approach is completely based on global analysis and Riemannian geometry. They first construct the space $\mathcal{C}_g$ of all complex structures on a smooth closed orientable surface of genus g by exploiting the fact that all almost complex structures are integrable in two dimensions. The goal is then to locally divide out the pullback action of the diffeomorphism group on $\mathcal{C}_g$ and obtain Teichmüller space as a manifold glued from local slices. In order to achieve this, a couple of specialties of surfaces have to be used, in particular the fact that one can identify $\mathcal{C}_g$ with the space of conformal classes of metrics, and that this space can be given a natural manifold structure again. But even with these tools at hand, the process of taking the slice is complicated by the fact that one works with spaces of smooth objects, which do not form Banach manifolds. Hence, one cannot use the implicit function theorem directly. The trick that circumvents this problem is to use Sobolev spaces throughout the whole work, and only in the end to show that everything remains true if one restricts oneself to smooth objects only.

It is clear that if one wants to carry some version of this procedure over to supergeometry, one will have to use infinite-dimensional supermanifolds. It is not enough to know how to handle infinite-dimensional super vector spaces, since they are *not* supermanifolds. Even with these tools at hand, Fischer and Tromba's approach cannot simply be repeated. First of all, supercomplex structures and superconformal structures do not necessarily coincide, and for $\mathfrak{k}^L(1|1)$-surfaces, they actually do not. Nonetheless, every super Riemann surface possesses a supercomplex structure, and it turns out that any deformation of its $\mathfrak{k}^L(1|1)$-structure also deforms the complex structure. Therefore, these two moduli problems cannot be separated, and one has to be treated as a subproblem of the other. Besides, it turns out that not all almost complex structures in $2|2$ real dimensions are integrable [Pol05]. So a suitble way to restrict the construction to the integrable ones is required. Another severe problem is the use of objects of finite differentiability class in Tromba's approach. In supergeometry, only smooth objects can usually be defined. It will be argued below that this restriction can, to a certain extent, be circumvented if one tailors special categories of superobjects. We could, however, not find a practicable way to employ these "C^k-supermanifolds" to the our problem. Luckily, it turns out that this is unnecessary because one can more or less separate the underlying moduli problem from its "odd parts", and the underlying one can be identified as a classical one. The topological and analytical subtleties pertain only to this underlying problem. But still, a full understanding of the diffeomorphism supergroup has to be achieved in order to be able to take a slice for its action. The investigation of the diffeomorphism supergroup and its

pullback action can indeed be seen as one of the main results of this work.

All this should make it clear that plenty of problems must be overcome to attempt a supergeometric version of Fischer and Tromba's approach. The resulting procedure only loosely follows their tracks, for the reasons outlined above. In particular, it requires many more algebraic considerations. Nonetheless, we finally arrive at local quotients for the action of the diffeomorphism supergroup which divide out as much of its action as is possible if one wants the quotient to be a supermanifold. The main result of this thesis is maybe not the super Teichmüller spaces themselves, which had been obtained by less technical means before. It is rather the tools and methods developed to pursue the global approach which should prove useful for many similar problems. Applications for such methods can, for example, be found in the study of classical configuration spaces of supersymmetric field theories, or for super Hilbert spaces and operators on them. In particular, the categorical approach can be expected to be a powerful tool if one tries to study supergeometric analogues of classical constructions. Examples are conformal and topological field theories, vertex operator superalgebras or automorphic forms, to name just a few.

1.4 Organization

Roughly a third of the work we present here is dedicated to a realization of the categorical programme set up by Molotkov in [Mol84], in particular to developing the part of his theory needed for our later applications together with complete proofs and examples. In Chapter 2, we give a brief overview of superalgebra and the ringed space formulation of supergeometry. Although this is rather standard material by now, there are only a small number of concise expositions, e.g., [Leiar], [DM99], [Var04].

In Chapter 3, the categorical approach is laid out in detail. The first sections contain a recapitulation of the necessary tools from homological algebra, like representable functors, inner Hom-objects and algebraic structures in categories.

In Section 3.3, we expound some of the consequences of the categorical approach, in particular the necessity of the use of families instead of single objects.

In Section 3.4, the reformulation of linear and commutative superalgebra in terms of the functor of points is given.

In Section 3.5 we return to geometry by defining superdomains of possibly infinite dimension. To be able to do so, we have to introduce a Grothendieck pretopology on a certain functor category which contains the functors of points for linear supermanifolds. This description allows the construction of Banach, Fréchet or even just locally convex superdomains. For the rest of the construction, we restrict ourselves to the Banach case in order not to overload this chapter with formalities. With some minor adaptions, everything goes through equally well for the Fréchet case.

In Section 3.6 we then define Banach supermanifolds as certain functors which

are locally isomorphic to Banach superdomains, along with supersmooth morphisms between them. The last section of Chapter 3 contains the definition of super vector bundles within the categorical framework.

Chapter 4 is the first one that is specialized to the application to superconformal geometry. We give a brief review over the classification of superconformal algebras which was obtained and expanded by many authors over the past decades. To only mention a small selection, contributions can be found in [ABD$^+$76b], [ABD$^+$76a], [NS71], [Ram71], [KvdL89], [SS87] and [GLS05].

We then identify the type of superconformal surface referred to as $N = 1$ super Riemann surface in the physics literature as the type associated with the algebra $\mathfrak{k}^L(1|1)$. A subsequent short analysis of the structure of general complex $1|1$-dimensional supermanifolds and $\mathfrak{k}^L(1|1)$-manifolds shows that the latter are subspecies of the former. While complex supermanifolds of dimension $1|1$ are equivalent to a Riemann surface together with a holomorphic line bundle on it, $\mathfrak{k}^L(1|1)$-surfaces are precisely those for which this line bundle is a spin bundle. This implies in particular that the moduli problem of super Riemann surfaces cannot be separated from that of supercomplex structures.

Chapters 5 and 6 are concerned with almost complex structures and their integrability on supermanifolds. They treat the general case and are not specialized to supersurfaces.

Chapter 5 contains two of our main results. The first is Thm. 5.1.1, which asserts that the functor of smooth sections of a super vector bundle (which is just the functor of points for the supermanifold of its sections) is represented by an infinite-dimensional super vector space. This intuitively plausible statement turns out to be rather hard to prove. The second result is the construction of a complex supermanifold $\mathcal{A}(\mathcal{M})$ of all almost complex structures on given supermanifold $\mathcal{M}$. Here, the power of the categorical framework becomes fully visible for the first time. We construct $\mathcal{A}(\mathcal{M})$ as a submanifold of the supermanifold of sections of the endomorphism bundle of the tangent bundle $\mathcal{TM}$ by adapting the method invented by Abresch and Fischer for ordinary almost complex manifolds [Tro92].

Chapter 6 is rather short, since we can base our analysis of integrability on the results of Vaintrob [Vai88a], [Vai85]. The main result is the determination of those deformations of an integrable almost complex structure which preserve integrability. Here, we follow a method which was proposed in a similar form by [GN88].

Chapter 7 contains another of the main results of this thesis, namely a complete description of the structure of the diffeomorphism supergroup $\widehat{\mathcal{SD}}(\mathcal{M})$ of a supermanifold. We first identify $\widehat{\mathcal{SD}}(\mathcal{M})$ as the subfunctor of the inner Hom-object $\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{M})$ consisting of invertible morphisms and show that it is in fact a restriction of $\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{M})$ to the group $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$.

In Section 7.2, we begin by analysing the "higher" functor points of $\widehat{\mathcal{SD}}(\mathcal{M})$, i.e., those which are not simply morphisms $\mathcal{M} \to \mathcal{M}$ but contain odd parameters.

It turns out that they have a very simple and neat description as exponentials of vector fields. The topological subtleties usually pertaining to the exponential in the case when the Lie group is not Banach only occur for the underlying group $\mathrm{Aut}(\mathcal{M})$. This allows us to split the superdiffeomorphism group $\widehat{\mathcal{SD}}(\mathcal{M})$ into a semidirect product of $\mathrm{Aut}(\mathcal{M})$ and a nilpotent group.

In Section 7.3 we complete the description of $\widehat{\mathcal{SD}}(\mathcal{M})$ by an analysis of $\mathrm{Aut}(\mathcal{M})$. The structure of this group is more intricate, but again it turns out that one can split it into several semidirect factors by comparing the supermanifold $\mathcal{M}$ to the exterior bundle $(M, \wedge^\bullet E)$ to which it is isomorphic by Batchelor's theorem [Bat79]. The deviation of $\mathcal{M}$ from the exterior bundle form can be expressed by an element of a nilpotent group $N_{\mathcal{M}}$ which is generated by the even nilpotent vector fields on $\mathcal{M}$. The remaining normal subgroup of $\mathrm{Aut}(\mathcal{M})$ can then be expressed as the isomorphisms of $(M, \wedge^\bullet E)$, which are those of a vector bundle E over the base manifold M. This group is again a semidirect product, namely of $\mathrm{Diff}(M)$, the diffeomorphisms of the underlying manifold, and $\mathrm{Aut}_M(E)$, the smooth automorphisms of E over M.

Only with this detailed description of $\widehat{\mathcal{SD}}(\mathcal{M})$ at hand does it become possible to try to find a slice for the pullback action of $\widehat{\mathcal{SD}}(\mathcal{M})$ on the integrable almost complex structures on $\mathcal{M}$, which would be a patch of super moduli space. As in the classical case, to divide out all of $\widehat{\mathcal{SD}}(\mathcal{M})$ at once is too ambitious, since the underlying Riemann surface of $\mathcal{M}$ may have additional nontrivial automorphisms. We confine ourselves to the identity component $\widehat{\mathcal{SD}}_0(\mathcal{M})$ to obtain the super Teichmüller space.

In Section 8.1.5 we determine the residual automorphisms contained in $\widehat{\mathcal{SD}}_0(\mathcal{M})$ and find out that there remains a nontrivial group of them. This will make it impossible to construct the Teichmüller space of $\mathfrak{vect}^L(1|1)$-surfaces as the base of a universal family. The supermanifold $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ that we construct instead still parametrizes all deformations of a given compact complex $1|1$-dimensional supermanifold. We did, however, not succeed in showing that it is the base of a semiuniversal family. We leave this problem for further investigations. The base manifold of $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ is the family $J(V_g)$ of Jacobian varieties over Teichmüller space constructed by Earle [Ear78]. This space can be seen as the Teichmüller space of pairs of Riemann surfaces together with holomorphic line bundles of a given fixed degree on them.

Finally we restrict this construction to $\mathfrak{k}^L(1|1)$-structures and we find that also in this case there exists no universal family (at least not in the category of supermanifolds). The reason is that there remains a residual group of automorphisms of the $\mathfrak{k}^L(1|1)$-surface which is isomorphic to $\mathbb{Z}_2$. This is in accord with previous results [LR88], [CR88].

1.5 Main results

The main results of this thesis can be summed up as follows.

1. We realize and extend the categorical programme laid out by Molotkov in [Mol84]. In particular, we present proofs to most of the statements claimed in [Mol84] and construct as an example for the viability of the categorical approach a manifold of all almost complex structures on a given almost complex supermanifold.

2. A detailed analysis of the structure of the diffeomorphism supergroup $\widehat{\mathcal{SD}}(\mathcal{M})$ of a smooth finite-dimensional supermanifold $\mathcal{M}$ is presented. This includes an investigation of the group $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$, as well as an analysis of the functor of points of $\widehat{\mathcal{SD}}(\mathcal{M})$. The latter is completely described in terms of a set of generators for the category of supermanifolds.

3. We present an anlysis of the deformations of two types of superconformal surfaces and investigate the viability of the global approach to Teichmüller theory of Tromba [Tro92] in the supergeometric context. We demonstrate that a universal Teichmüller family does not exist for compact complex $1|1$-dimensional supermanifolds, but that a semiuniversal one can be constructed for $N = 1$ super Riemann surfaces, i.e., $\mathfrak{k}^L(1|1)$-surfaces.

Except for [Mol84], infinite-dimensional supergeometry has, to our knowledge, only been thoroughly investigated in [Sch97]. The latter, however, takes a quite different route. Bits and pieces of the categorical approach have appeared throughout the literature, e.g., the so-called "even-rules principle" [DM99] or the quite well-known functor of points. The full program proposed in [Mol84] has, however, never been realized and applied before.

A full investigation of the structure of the diffeomorphism supergroup has seemingly also not been given before. The group $\mathrm{Aut}(\mathcal{M})$ has, of course, been mentioned in many places, but the difficulties of its infinite dimensionality seem to have obstructed its analysis.

The deformation theory of complex superspaces has been developed in great detail by Vaintrob [Vai88a], [Vai86], [Vai88b]. In particular, he proved the existence of versal deformations in the complex $1|1$-dimensional case. In a similar way, Flenner and Sundararaman investigated deformations of complex analytic superspaces [FS92]. The Teichmüller space of $\mathfrak{k}^L(1|1)$-surfaces was investigated by Crane and Rabin [CR88], Rothstein and LeBrun [LR88], Hodgkin [Hod95], [Hod87], Nelson and Giddings [GN88], and many others. The authors of [CR88] observed that no universal family of marked $\mathfrak{k}^L(1|1)$-surfaces exists. Rothstein and LeBrun consequently divided out the resulting ambiguities and introduced for the description of the resulting space the concept of a superorbifold. Natanzon in [Nat04] developed an approach to the moduli problems of $N = 1$ and $N = 2$ super Riemann surfaces based on superanalogs of algebraic curves and Fuchsian groups. This approach uses completely different tools than the ones employed in this thesis, but on the other hand the construction of the moduli spaces instead of only the Teichmüller spaces, as stated in [Nat04], would be quite an improvement as compared to our results. A comparison of the results of Natanzon with ours,

however, would have exceeded the scope of this work by far. We must leave it as an interesting topic for future research.

The contribution of this thesis to super Teichmüller theory is twofold. On one hand we investigate the possibility of a global approach to supergeometric moduli problems. This should be understood also as a methodological result, since global techniques may prove valuable for similar problems, e.g., the study of classical configuration spaces of supersymmetric gauge theories or variational problems in supergeometry. On the other we study here (to our knowledge for the first time) the Teichmüller space of general complex $1|1$-dimensional supermanifolds and determine the obstructions to the construction of a universal family of those.

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[1] Despite me not always listening carefully enough!

Chapter 2

Supergeometry

This chapter is dedicated to an outline of the general theory of supergeometry, as far as it will be needed in the following chapters. Here, we first present the ringed space formalism of supergeometry. The categorical approach will be presented in a separate chapter. Detailed accounts of the materials of this chapter can be found in [DM99],[Var04].

Roughly speaking, supersymmetry deals with $\mathbb{Z}_2$-graded objects [1] and morphisms between them which preserve the grading. It might seem somewhat arbitrary to restrict oneself to the case of just $\mathbb{Z}_2$-grading, and not to consider gradings by arbitrary abelian groups. But it has turned out that this case plays a special role: extending the scope beyond the linear realm into geometry, the $\mathbb{Z}_2$-grading becomes translated into commuting and anticommuting variables. Thus, supergeometry is a special kind of noncommutative geometry, but its noncommutativity is one of the tamest possible ones. While graded objects had been used in algebra for a long time already, Berezin's idea to construct a theory of analysis and geometry which uses commuting and anticommuting variables on the same footing was quite revolutionary. As could be expected, this geometry is considerably more algebraic than classical differential geometry. In particular, the use of non-reduced ringed spaces for the definition of supermanifolds becomes inevitable.

2.1 Linear superspaces and superalgebras

Throughout this work, we denote the elements of $\mathbb{Z}_2$ as $\{\bar{0}, \bar{1}\}$. The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. *A ring R is called a superring, if it is $\mathbb{Z}_2$-graded, i.e., if it decomposes into a direct sum of additive subgroups $R = R_{\bar{0}} \oplus R_{\bar{1}}$ for which*

$$R_{\bar{i}} \cdot R_{\bar{j}} \subseteq R_{\bar{i}+\bar{j}} \tag{2.1}$$

[1] By $\mathbb{Z}_2$, we denote the field of residues $\mathbb{Z}/2\mathbb{Z}$. The 2-adic numbers, which are sometimes denoted by the same symbol, do not occur in this work.

holds. A module M over a superring R is called a supermodule, if it is $\mathbb{Z}_2$-graded,

$$M = M_{\bar{0}} \oplus M_{\bar{1}}, \tag{2.2}$$

and if $R_{\bar{i}} \cdot M_{\bar{j}} \subseteq M_{\bar{i}+\bar{j}}$. The submodule $M_{\bar{0}}$ is called even, $M_{\bar{1}}$ is called odd. A morphism $\phi : M \to M'$ of R-supermodules is a morphism of R-modules which preserves the grading.

We denote by SMod_R (resp. ${}_R\mathsf{SMod}$) the category of right (resp. left) R-supermodules. Note that any ring R can be considered as a superring by just setting $R_{\bar{0}} := R$ and $R_{\bar{1}} := 0$.

Definition 2.1.2. *Let M be an R-supermodule. An element $m \in M$ is called homogeneous, if $m \in M_{\bar{0}}$ or $m \in M_{\bar{1}}$. For a homogeneous element, its parity is defined to be*

$$p(m) := \begin{cases} 0 & \text{if} \quad m \in M_{\bar{0}} \\ 1 & \text{if} \quad m \in M_{\bar{1}} \end{cases} \tag{2.3}$$

An inhomogeneous element is said to be of indefinite parity.

Definition 2.1.3. *A $\mathbb{K}$-supermodule is called a $\mathbb{K}$-super vector space.*

The category of $\mathbb{K}$-super vector spaces will be denoted by $\mathsf{SVec}_{\mathbb{K}}$. Particularly important are the standard super vector spaces $\mathbb{K}^{m|n}$, which have m even and n odd dimensions.

Definition 2.1.4. *A $\mathbb{K}$-superalgebra A is a $\mathbb{K}$-super vector space endowed with a morphism*

$$\mu : A \otimes_{\mathbb{K}} A \to A. \tag{2.4}$$

The algebra A is called supercommutative if for all homogeneous elements a, b, one has

$$\mu(a, b) = (-1)^{p(a)p(b)}\mu(b, a). \tag{2.5}$$

A is associative, resp. with unit, if it is associative, resp. has a unit, as an ordinary $\mathbb{K}$-algebra.

Definition 2.1.5. *A Lie superalgebra L over $\mathbb{K}$ is a $\mathbb{K}$-superalgebra whose morphism $[\cdot, \cdot] : L \otimes L \to L$ satisfies the following properties:*

1. *it is super-antisymmetric: $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ for all homogeneous elements $a, b \in L$,*

2. *it satisfies the super Jacobi identity, i.e., for all homogeneous $a, b, c \in L$, one has*

$$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0. \tag{2.6}$$

Expressions like (2.5) and (2.6) are extended to inhomogeneous elements by linearity. It is clear that if a unit exists, it has to be even. If no confusion can arise, multiplication in an algebra will be denoted by either a dot (like $a \cdot b$) or by juxtaposition (like ab). In the following, we will assume that all superalgebras are associative and have a unit.

Definition 2.1.6. *A left (resp. right) module over a $\mathbb{K}$-superalgebra A is a $\mathbb{K}$-super vector space M endowed with a morphism*

$$\rho : A \otimes_{\mathbb{K}} M \to M \qquad (\textit{resp. } \rho : M \otimes_{\mathbb{K}} A \to M). \tag{2.7}$$

satisfying the usual identities that make M a module over A as an ordinary algebra.

For a supercommutative algebra, every left module can be made a right module by defining

$$m \cdot a := (-1)^{p(a)p(m)} a \cdot m \tag{2.8}$$

for all homogeneous elements $a \in A$ and $m \in M$, and extending this definition by linearity. From now on, we will restrict ourselves to supercommutative algebras with unit. Consequently, we will just speak of "modules", all of which are defined to be left-modules whose right module structure is given by (2.8).

Definition 2.1.7. *Let A be a supercommutative superalgebra. The direct sum of A-super modules $M = M_{\bar{0}} \oplus M_{\bar{1}}$ and $N = N_{\bar{0}} \oplus N_{\bar{1}}$ is a supermodule whose homogeneous submodules are given by*

$$(M \oplus N)_{\bar{i}} = M_{\bar{i}} \oplus N_{\bar{i}}. \tag{2.9}$$

This definition extends to direct sums of families indexed by a set $\mathcal{I}$ in the obvious way.

Actually, Def. 2.1.7 is a consequence of the general definition of direct sums as colimits of certain functors (cf. Section 3.1.4).

2.1.1 Tensor products of supermodules

Definition 2.1.8. *Let M, N be $\mathbb{K}$-super vector spaces. The tensor product of M and N is the tensor product of M, N as ordinary $\mathbb{K}$-super vector spaces, endowed with the grading*

$$(M \otimes_{\mathbb{K}} N)_{\bar{i}} = \bigoplus_{\bar{j}+\bar{k}=\bar{i}} M_{\bar{j}} \otimes_{\mathbb{K}} N_{\bar{k}}. \tag{2.10}$$

If A, B are $\mathbb{K}$-superalgebras, their tensor product can be given a canoncial $\mathbb{K}$-superalgebra structure again by setting

$$(a \otimes b)(c \otimes d) := (-1)^{p(b)p(c)} ac \otimes bd. \tag{2.11}$$

This extends to modules over supercommutative superalgebras in a natural way.

Definition 2.1.9. *Let M, N be modules over a $\mathbb{K}$-superalgebra A. Their tensor product is defined as*

$$(M \otimes_A N) := (M \otimes_{\mathbb{K}} N)/I, \tag{2.12}$$

where I is the ideal spanned by elements of the form $ma \otimes n - m \otimes an$, $m \in M, n \in N, a \in A$.

The definition of the tensor product is chosen such that it has "good" properties. The tensor product of modules over some commutative ring R has several functorial properties, e.g. universality in the category of multilinear maps. Besides, it comes equipped with a unit object (R itself) and an associativity isomorphism. Categories which allow a product which has these properties are called *tensor categories*. Usually one supplements this by a commutativity isomorphism $c_{V,W} : V \otimes W \cong W \otimes V$. If the isomorphisms $c_{V,W}$ satisfy certain compatibility axioms with respect to associativity and the unit element[2], they are called a *braiding* of the tensor category. The category of supermodules over a supercommutative $\mathbb{K}$-superalgebra A inherits all this from the tensor product in the category of modules over A as an ordinary algebra, except for one crucial difference: it has a different braiding, namely [DM99]

$$\begin{aligned} c_{V,W} : V \otimes_A W &\to W \otimes_A V \\ v \otimes w &\mapsto (-1)^{p(v)p(w)} w \otimes v. \end{aligned} \tag{2.13}$$

If one has several factors arranged in two different permutations $V = V_1 \otimes \ldots \otimes V_n$ and $V' = V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$, the axioms of a braided tensor category require that there exists a unique isomorphism $\tau : V \cong V'$. This isomorphism is given by a composition of the associativity and commutativity isomorphisms of the factors, and the axioms thus require that whichever composition we choose, the result is the same. Since we have inherited associativity from the category of ordinary modules and commutativity is given by (2.13), τ must be the map

$$\tau : v_1 \otimes \ldots \otimes v_n \mapsto (-1)^N v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}, \tag{2.14}$$

where N is the number of pairs of indices where $i < j$ but $\sigma(i) > \sigma(j)$, and for which v_i, v_j are odd. This fact is the origin of the *sign rule*, which says that whenever one interchanges two neighbouring odd factors in a product of elements of some superalgebra, one picks up a factor (-1).

Definition 2.1.10. *Let R be a superring. The change of parity functor is the functor $\Pi : \mathsf{SMod}_R \to \mathsf{SMod}_R$ (equivalently for left modules) which assigns to a supermodule $M = M_{\bar{0}} \oplus M_{\bar{1}}$ the supermodule $\Pi(M)$ with $(\Pi(M))_{\bar{0}} = M_{\bar{1}}$ and $(\Pi(M))_{\bar{1}} = M_{\bar{0}}$.*

Parity reversal has to be a functor since any morphism between two supermodules has to preserve parity. One clearly has $\Pi(\mathbb{K}) = \mathbb{K}^{0|1}$. With the help

[2] The so-called Hexagon axioms, see, e.g., [BK01]

of definition (2.1.8), one also easily verifies the useful fact that for any $\mathbb{K}$-super vector space V,

$$\mathbb{K}^{0|1} \otimes_{\mathbb{K}} V \cong \Pi(V). \tag{2.15}$$

2.1.2 $\mathbb{Z}$-graded supermodules

Some constructions yield modules and spaces which are already naturally $\mathbb{Z}$-graded, e.g., exterior or symmetric algebras over super vector spaces, or the complex of differential forms appearing below. One has to be careful to choose a consistent convention for the interplay of the degree with parity. There are two choices [DM99], and although they are basically equivalent, they lead to sign differences in some formulas. To remove all ambiguities it is sufficient to fix the commutativity isomorphism of the tensor product. Following [DM99], we choose the following one.

Definition 2.1.11. *Let A be a commutative $\mathbb{K}$-superalgebra. Define a tensor category SMod_A^{gr} of $\mathbb{Z}$-graded objects of SMod_A by defining the tensor product of any two $\mathbb{Z}$-graded A-supermodules M, N to be the tensor product $M \otimes_A N$ in SMod_A with the usual associativity isomorphism, but with the commutativity isomorphism redefined as*

$$\begin{aligned} c_{M,N} : M \otimes_A N &\rightarrow N \otimes_A M & (2.16)\\ m \otimes n &\mapsto (-1)^{p(m)p(n)+\deg(m)\deg(n)} n \otimes m. & (2.17)\end{aligned}$$

This redefines the $c_{M,N}$ of SMod_A to $(-1)^{\deg(m)\deg(n)} c_{M,N}$ for all $m \in M$ and $n \in N$.

An application of this rule is the construction of the symmetric and exterior powers of a super vector space V. The symmetric power $\mathrm{Sym}^n(V)$ is the quotient of $V \otimes \ldots \otimes V$ by the action of the commutativity isomorphisms $c_{V,V} : V \otimes V \to V \otimes V$ exchanging the elements of all factors in combination with associativity. Clearly one has

$$\mathrm{Sym}^\bullet(V) = \mathrm{Sym}^\bullet(V_{\bar{0}}) \otimes \wedge^\bullet(V_{\bar{1}}) \tag{2.18}$$

where Sym and $\wedge$ on the right hand side are the operations in the category of ordinary vector spaces.

Analogously the exterior power $\wedge^n(V)$ of a super vector space V is the quotient of $V \otimes \ldots \otimes V$ by the action of the symmetric group S_n acting via $c_{V,V}$ on the factors, additionally multiplying the permutation with its sign character $\epsilon(\sigma)$. The result is

$$\wedge^\bullet (V) = \wedge^\bullet(V_{\bar{0}}) \otimes \mathrm{Sym}^\bullet(V_{\bar{1}}). \tag{2.19}$$

Note that as an object of $\mathsf{SMod}_{\mathbb{K}}^{gr}$, $\wedge^\bullet(V)$ is the symmetric algebra of the object $V \in \mathsf{SMod}_{\mathbb{K}}^{gr}$ which is purely of degree one.

2.1.3 The inner Hom object for supermodules

If V, W are two $\mathbb{K}$-vector spaces, the set of their morphisms, i.e. linear maps $V \to W$, naturally carries the structure of a $\mathbb{K}$-vector space itself. The same goes for, e.g., abelian groups, sets and many other categories. This situation is axiomatized by the so-called inner Hom object [Mac98],[GM02]. Let C be some category with a product operation $*$, e.g. Sets with the direct product, or $\mathbb{K}$-vector spaces with the tensor product over $\mathbb{K}$. Then, for $Y, Z \in \mathsf{C}$, the inner Hom object $\underline{\mathrm{Hom}}(Y, Z)$ is an object satisfying the *adjunction formula* [GM02]:

$$\mathrm{Hom}_{\mathsf{C}}(X, \underline{\mathrm{Hom}}(Y, Z)) = \mathrm{Hom}_{\mathsf{C}}(X * Y, Z) \qquad \forall X \in \mathsf{C}. \tag{2.20}$$

This formula requires that $\underline{\mathrm{Hom}}(Y, Z)$, if it exists, must represent the functor

$$\begin{aligned} \mathsf{C}^\circ &\to \mathsf{Sets} &(2.21)\\ X &\mapsto \mathrm{Hom}(X * Y, Z), &(2.22) \end{aligned}$$

where C° denotes the dual category of C. This implies that if $\underline{\mathrm{Hom}}(Y, Z)$ exists, it is unique up to a unique isomorphism. In view of the above examples, the interpretation is clear: the inner Hom object $\underline{\mathrm{Hom}}(Y, Z)$ is intended to be an object which turns the application of a map $\varphi : Y \to Z$ into an *operation* $\underline{\mathrm{Hom}}(Y, Z) * Y \to Z$ in C.[3]

In the super context, inner Hom objects play an important role as superizations of spaces of maps between super objects. In this work they will be needed for various $\mathbb{K}$-super modules and for supermanifolds (e.g., as the diffeomorphism supergroup).

As an example, look at $\mathbb{K}$-super vector spaces first. For V, W two $\mathbb{K}$-super vector spaces the set $\mathrm{Hom}(V, W)$ naturally inherits the structure of a $\mathbb{K}$-vector space but it is not a super vector space: there is no natural grading. The inner Hom object will have to satisfy

$$\mathrm{Hom}_{\mathsf{SVec}_{\mathbb{K}}}(U, \underline{\mathrm{Hom}}(V, W)) = \mathrm{Hom}_{\mathsf{SVec}_{\mathbb{K}}}(U \otimes_{\mathbb{K}} V, W) \tag{2.23}$$

for all $\mathbb{K}$-super vector spaces U. Since $\mathbb{K}^{1|0}$ and $\mathbb{K}^{0|1}$ generate the category $\mathsf{SVec}_{\mathbb{K}}$ (cf. Prop. 3.1.4), it is enough to let U run over these two spaces. One finds

$$\mathrm{Hom}(\mathbb{K}^{1|0} \otimes V, W) \cong \mathrm{Hom}(V, W) \tag{2.24}$$

and

$$\mathrm{Hom}(\mathbb{K}^{0|1} \otimes V, W) \cong \mathrm{Hom}(\Pi(V), W). \tag{2.25}$$

Thus the even part $\underline{\mathrm{Hom}}(V, W)_{\bar{0}}$ consists of the $\mathbb{K}$-linear maps $V \to W$ which preserve the grading (i.e. the actual morphisms of $\mathsf{SVec}_{\mathbb{K}}$), while $\underline{\mathrm{Hom}}(V, W)_{\bar{1}}$ consists of those which reverse it. Elements of the even part are therefore called

[3] A (left) operation of an object K on an object A is a morphism $K \times A \to A$.

even linear maps, those of the odd part odd linear maps. One should, however, bear in mind that only the even ones are really morphisms in the strict sense.

In a similar way [DM99] one shows that for modules M, N over a supercommutative super algebra A the inner Hom object $\underline{\mathrm{Hom}}_A(M,N)$ is the $\mathbb{K}$-super vector space of even, resp. odd, maps $f : M \to N$ for which

$$f(am) = (-1)^{p(f)p(a)} a f(m). \tag{2.26}$$

This enforces $f(ma) = f(m)a$. The A-module structure is given by

$$(af)(m) = af(m). \tag{2.27}$$

Thus again, $\underline{\mathrm{Hom}}_A(M,N)_{\bar{0}} = \mathrm{Hom}_A(M,N)$. This turns out to be a general feature of inner Hom objects in categories of super objects: their underlying space is always the set of actual morphisms between the two objects in question. The parts of them which involve odd elements are only made visible via products with other super objects.

2.1.4 Derivations of superalgebras

A derivation of a $\mathbb{K}$-superalgebra A is a morphism $D : A \to A$ such that

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b). \tag{2.28}$$

By this definition, the set of derivations would only be an ordinary Lie algebra over $\mathbb{K}$. To make it a Lie superalgebra, one has to allow odd derivations, i.e., odd elements of $\underline{\mathrm{Hom}}_{\mathbb{K}}(A,A)$.

Definition 2.1.12. *Let A be a $\mathbb{K}$-superalgebra. The Lie superalgebra $\underline{\mathrm{der}}_{\mathbb{K}}(A,A)$ of derivations of A is the subsuperspace of $\underline{\mathrm{Hom}}_{\mathbb{K}}(A,A)$ whose elements D satisfy*

$$D(a \cdot b) = (Da) \cdot b + (-1)^{p(D)p(a)} a \cdot Db. \tag{2.29}$$

One easily checks that this defines a Lie super algebra, with bracket

$$[D_1, D_2] = D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1. \tag{2.30}$$

2.1.5 Dual modules

In a tensor category with inner Hom objects, one defines the dual object of M as [DM99]

$$M^* := \underline{\mathrm{Hom}}(M, \underline{1}) \tag{2.31}$$

where $\underline{1}$ is the unit of the tensor product. In the case of modules over a supercommutative superalgebra A one has $\underline{1} = A$. Using the adjunction formula (2.20), this yields a canonical morphism

$$M^* \otimes_A N \quad \to \quad \underline{\mathrm{Hom}}(M,N) \tag{2.32}$$

$$\omega \otimes n \quad \mapsto \quad \Big(m \mapsto (-1)^{p(\omega)p(n)} n\omega(m)\Big), \tag{2.33}$$

which is an isomorphism if M is free of finite type [DM99]. Dual modules appear naturally in the context of vector bundles on super manifolds.

2.1.6 The category of finite-dimensional Grassmann algebras

A particularly important role will be played by the finite dimensional Grassmann algebras Λ_n. They are the free commutative superalgebras on n odd generators. Equivalently, they can be thought of as the exterior algebras of $\mathbb{K}$-vector spaces of $\mathbb{K}$-dimension n. To make it precise, they are generated over $\mathbb{K}$ by generators $\xi_1, \ldots, \xi_n$ subject to the relations

$$\xi_i \xi_j = -\xi_j \xi_i, \qquad \text{for } i, j \in \{1, \ldots, n\}. \tag{2.34}$$

Together with their morphisms as superalgebras, the finite-dimensional Grassmann algebras form a category Gr. This category has a terminal object: the algebra $\Lambda_0 = \mathbb{K}$. A terminal object has the property that every other object possesses a unique arrow to it, called the terminal morphism. In the case of Gr, this morphism acts by removing all odd generators:

$$\begin{aligned} \epsilon_{\Lambda_n} : \Lambda_n &\rightarrow \mathbb{K} \\ 1 &\mapsto 1 \\ \xi_i &\mapsto 0, \qquad \text{for } 1 \leq i \leq n. \end{aligned} \tag{2.35}$$

Note that $\Lambda_0 = \mathbb{K}$ is also the initial object of Gr: there is a unique morphism

$$c_{\Lambda_n} : \mathbb{K} \rightarrow \Lambda_n \qquad \text{for all } n \in \mathbb{N} \tag{2.36}$$

which embeds the ground field into Λ_n by sending 1 to 1. Thus $\mathbb{K}$ is the *null object* of Gr.

We will write Gr only for the real Grassmann algebras, i.e. those with $\mathbb{K} = \mathbb{R}$. In the case of $\mathbb{K} = \mathbb{C}$ we denote the appropriate category as $\mathsf{Gr}^{\mathbb{C}}$, to avoid confusion. The complex algebras will be denoted $\Lambda_n^{\mathbb{C}}$. As exterior algebras they are generated over complex vector spaces of dimension n, but it is easy to show that one gets the same if one defines them as complexifications of the real Grassmann algebras. More precisely, if V is a real vector space, then

$$\Lambda(V \otimes_{\mathbb{R}} \mathbb{C}) \cong \Lambda(V) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.37}$$

Proposition 2.1.13. *Let Λ_n, Λ_m be the Grassmann algebras over $\mathbb{K}$ on n and m generators, respectively. Then there exists an isomorphism of $\mathbb{K}$-vector spaces*

$$\mathrm{Hom}_{\mathsf{Gr}}(\Lambda_n, \Lambda_m) \cong \mathbb{K}^n \otimes_{\mathbb{K}} \Lambda_{m,\bar{1}}. \tag{2.38}$$

Proof. Let $\xi_1, \ldots, \xi_n$ be the free generators of Λ_n. A morphism $\phi : \Lambda_n \rightarrow \Lambda_m$ is a homomorphism of $\mathbb{K}$-algebras which preserves parity. Thus $\phi(1) = 1$, and ϕ is uniquely determined by choosing the images $\phi(\xi_n)$ of its generators which have to lie in $\Lambda_{m,\bar{1}}$. Setting $V = \mathrm{Span}_{\mathbb{K}}(\xi_1, \ldots, \xi_n)$, we can write

$$\mathrm{Hom}_{\mathsf{Gr}}(\Lambda_n, \Lambda_m) \cong \Lambda_{m,\bar{1}} \otimes_{\mathbb{K}} V^*,$$

where V^* is the dual space of V, which is in our case isomorphic to $\mathbb{K}^n$. $\square$

2.2 Supermanifolds

Finite-dimensional supermanifolds are most conveniently defined using the language of ringed spaces. Although it is possible to give a description in terms of charts and atlases, the ringed space formalism is an indispensable tool. Furthermore, if one wants to describe singular superspaces, the ringed space approach seems to be the only viable one. The description of supermanifolds resembles in this regard the theory of complex analytic spaces [GR84] (where non-reduced spaces also appear). However, there are important deviations from ordinary geometry, mainly originating in the fact that one would like to retain the concept of a coordinate for something that one calls a manifold, but coordinates cannot play the same role as usual in the context of nilpotent elements. In particular, it turns out that one must be careful with the notion of a "function" on a supermanifold. While a function on an ordinary manifold is described by the values it takes at every point (and thus coordinates are just a suitable collection of functions whose values label the points), such an interpretation cannot directly be carried over the super context. Functions considered as maps from the supermanifold to the ground field $\mathbb{K}$ would not form a superalgebra; they only inherit the structure of a $\mathbb{K}$-algebra. To make the full geometric structure visible in terms of maps it is therefore often necessary not to consider single superringed spaces but families of them over a base which is a supermanifold itself. The use of such families has always been common in physics, but often implicit, by using "Grassmann-valued" functions. We will argue below that this method, although it often yields the correct results when carefully used, is not really clean. Rather one should clearly distinguish between single supermanifolds and families of them. Another (and mathematically rigorous) motivation for the use of families instead of single objects will be supplied by the functor of points approach in the next chapter.

2.2.1 Superspaces

Definition 2.2.1. *A locally superringed space $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ is a topological space M endowed with a sheaf $\mathcal{O}_\mathcal{M}$ of local supercommutative rings. A morphism of superspaces is a morphism of locally ringed spaces which is stalkwise a homomorphism of supercommutative rings.*

The structure sheaf $\mathcal{O}_\mathcal{M}$ contains a canonical subsheaf of nilpotent ideals generated by odd elements:

$$\mathcal{J} = \mathcal{O}_{\mathcal{M},\bar{1}} \oplus \mathcal{O}^2_{\mathcal{M},\bar{1}}. \tag{2.39}$$

The superspace $\mathcal{M}_{red} = (M, \mathcal{O}_\mathcal{M}/\mathcal{J})$ is then purely even and if there are no even nilpotent elements in $\mathcal{O}_\mathcal{M}$, it is also reduced. It possesses a canonical embedding

$$\mathrm{cem} : \tilde{\mathcal{M}} \to \mathcal{M}. \tag{2.40}$$

The space $\mathcal{M}_{red}$ is called the underlying space of $\mathcal{M}$. In general, $\mathcal{M}_{red}$ can itself be a non-reduced space. The "completely reduced" space is denoted as $\mathcal{M}_{rd}$. In

the structure sheaves of the superspaces considered in this work no nilpotent even elements occur, so no confusion can arise. In this case, we speak of $\mathcal{M}_{red} = \mathcal{M}_{rd}$ as the underlying manifold.

2.2.2 Superdomains and supermanifolds

As in ordinary algebraic geometry, a supermanifold is a superspace which has only smooth points. The notion of "smooth point", however, is somewhat different in the super context, as it still allows nilpotent elements in the stalks. But still, a supermanifold is defined by the property that each of its points has a neighbourhood which is isomorphic to the same model space.

Definition 2.2.2. *Let V be a real super vector space of dimension $m|n$. To V we associate a locally ringed space $\overline{V}$ by setting*

$$\overline{V} := (V_{\bar{0}}, C^\infty_{V_{\bar{0}}} \otimes_{\mathbb{R}} \wedge[\theta_1, \ldots, \theta_n]), \tag{2.41}$$

where $C^\infty_{V_{\bar{0}}}$ denotes the sheaf of smooth functions, and $\theta_1, \ldots, \theta_n$ are a basis of $V_{\bar{1}}$. $\overline{V}$ is called the smooth linear supermanifold associated with V.

Together with their morphisms as superringed spaces linear supermanifolds form a category. An isomorphism $V \to V$ corresponds to an isomorphism $\overline{V} \to \overline{V}$ of ringed spaces, therefore it is sufficient to consider the equivalent full subcategory $\overline{\mathbb{K}^{m|n}}$ for $m, n \in \mathbb{N}_0$.

Definition 2.2.3. *A smooth superdomain $\mathcal{U}$ of dimension $m|n$ is a restriction of $\overline{\mathbb{R}^{m|n}}$ to an underlying domain $U \subseteq \mathbb{R}^m$:*

$$\mathcal{U} = (U, C^\infty_U \otimes_{\mathbb{R}} \wedge[\theta_1, \ldots, \theta_n]\big|_U). \tag{2.42}$$

Definition 2.2.4. *A smooth supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ of dimension $m|n$ is a superspace which is locally isomorphic to a linear supermanifold of dimension $m|n$.*

Definition 2.2.5. *An open subsupermanifold $\mathcal{U}$ of $\mathcal{M}$ is defined by an open subset U of M and the restriction of $\mathcal{O}_{\mathcal{M}}$ to U:*

$$\mathcal{U} = (U, \mathcal{O}_{\mathcal{M}}\big|_U). \tag{2.43}$$

Analogously, one defines real analytic superdomains as open subsupermanifolds of the linear real analytic supermanifold $(\mathbb{R}^m, \mathcal{A}(\mathbb{R}^m)[\theta_1, \ldots, \theta_n])$, and real analytic supermanifolds as being locally isomorphic to them. For complex analytic supermanifolds, one first constructs the reference space

$$\overline{\mathbb{C}^{m|n}} = (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m} \otimes_{\mathbb{C}} \wedge[\theta_1, \ldots, \theta_n]), \tag{2.44}$$

where $\mathcal{O}(\mathbb{C}^m)$ denotes the sheaf of holomorphic functions. A complex (analytic) supermanifold of dimension $m|n$ is then defined by the property of being locally isomorphic to $\overline{\mathbb{C}^{m|n}}$.

It is important to note that, unlike in the purely even case, a super vector space is *not* a supermanifold. Although there exists a canonical way to produce a linear supermanifold $\overline{V}$ from any finite-dimensional super vector space V the two objects behave quite differently. For example, if $\dim_{\mathbb{R}} V = m|n$, the set-theoretical model underlying V is a linear $m+n$-dimensional space, while that of $\overline{V}$ is only an m-dimensional space.

Together with their morphisms as superringed spaces, smooth finite-dimensional supermanifolds form a category which we will denote as $\mathsf{FinSMan}$. For the category of complex supermanifolds, whose morphisms are those of topological spaces ringed with local $\mathbb{C}$-superalgebras, we write $\mathsf{FinSMan}^{\mathbb{C}}$.

An example for a morphism of supermanifolds is the embedding of its underlying manifold. The nilpotent elements of each stalk form an ideal in this stalk and these together form an ideal subsheaf $\mathcal{J} \subset \mathcal{O}_{\mathcal{M}}$ (cf. (2.39)). Then one has the canonical epimorphism $q : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}/\mathcal{J}$ of sheaves, and setting

$$\mathrm{cem} = (\mathrm{id}_M, q) : M_{rd} \quad \to \quad \mathcal{M}, \tag{2.45}$$

one obtains a canonical embedding of M_{rd} as a closed subsupermanifold of $\mathcal{M}$. Note that in general there is no canonical projection $\mathrm{cpr} = (\mathrm{id}_M, \phi) : \mathcal{M} \to M_{rd}$. To construct such a projection, we need a sheaf morphism $\phi : \mathcal{O}_{\mathcal{M}}/J \to \mathcal{O}_{\mathcal{M}}$. Of course, such morphisms always exist, but they are not canonically given. An exception is the case of a split supermanifold (see below), i.e., a supermanifold which is globally isomorphic to the exterior bundle of a vector bundle over M_{rd}.

2.2.3 Sections, functions, and values

One refers to the sections of the structure sheaf as “superfunctions”, in analogy with the case of a purely even manifold. One has to be careful with this intuition, however. The sections of a sheaf of $\mathbb{K}$-superalgebras cannot straightforwardly be interpreted as maps from the supermanifold $\mathcal{M}$ to some other (super)space, in particular not to the field $\mathbb{K}$.

In the case of an ordinary manifold M, even if is initially defined abstractly just by a sheaf $\mathcal{O}_M$ of commutative $\mathbb{K}$-algebras with unit, one can reinterpret the sections of $\mathcal{O}_M$ as actual functions, i.e. as local morphisms $M \to \mathbb{K}$. Namely one ascribes to each section $s \in \mathcal{O}_M(U)$ and each point $p \in U$ its *value* $\lambda = s(p)$, which is defined as the unique number in $\mathbb{K}$, such that $s - \lambda$ is not invertible in any neighbourhood of p. In the case of a reduced space the values $s(p)$ for all $p \in U$ determine s uniquely. Conversely, this fact enables one to choose n suitable coordinate functions, whose values can be used to label the points of M. This method does not extend to nonreduced spaces [GM02],[GR84], thus in particular not to superspaces.

Definition 2.2.6. *Let $\mathcal{M}$ be a supermanifold (real or complex). Let $U \subseteq M$ be an open set of M and let $f \in \mathcal{O}_{\mathcal{M}}(U)$ be a section of the structure sheaf over U.*

Then we define the evaluation of f at some point $p \in M$ as the map

$$\mathrm{ev}_p : f \mapsto f_{red}(p), \tag{2.46}$$

where f_{red} is the image of f under the sheaf map $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}/\mathcal{J}$.

Evaluation of functions is a sheaf homomorphism which is equivalent to reducing the structure sheaf. So it makes sense to speak of the values of a section of $\mathcal{O}_{\mathcal{M}}$ for a supermanifold $\mathcal{M}$ and the value at p is still the unique number λ in $\mathbb{K}$ such that $s - \lambda$ is not invertible in any neighbourhood of p. It just does not specify s uniquely — for example a purely odd section will have the value zero everywhere, but need not be the zero section. The fact that its values don't characterise a superfunction uniquely also carries over to sections of modules over the structure sheaf, such as vector fields and differential forms.

2.2.4 Coordinates

Definition 2.2.7. *Let $\mathcal{M}$ be a smooth supermanifold, and let $\Phi = (\phi, \varphi) : \mathcal{M} \to \mathcal{V} \subseteq \overline{\mathbb{R}^{m|n}}$ be a morphism to a superdomain. The coordinates of Φ are defined to be the pullback of the standard coordinates $(x_1, \ldots, x_m, \theta_1, \ldots, \theta_n)$ of $\overline{\mathbb{R}^{m|n}}$ to $\mathcal{M}$, i.e. they are*

$$\begin{aligned} y_i &= \varphi(x_i) & (2.47)\\ \eta_j &= \varphi(\theta_j). & (2.48) \end{aligned}$$

As remarked in the previous section, the idea to describe the points of a manifold in terms of tuples of real (or complex) numbers which "label" its topological points is inadequate for nonreduced spaces. In the theory of complex analytic spaces [GR84] one just drops this kind of intuition and resorts to the algebraic description. For supermanifolds, however, it is possible to maintain the "coordinate" concept in the sense that one may still describe them locally in terms of standard (super)functions on model domains which one pulls back along an isomorphism. The crucial statement that makes this construction work is:

Theorem 2.2.8. *Let $\mathcal{V} \subseteq \overline{\mathbb{R}^{m|n}}$ be a superdomain and $\mathcal{M}$ be an arbitrary supermanifold. Then the map which assigns to each morphism $f : \mathcal{M} \to \mathcal{V}$ its coordinates $(y_1, \ldots, y_m, \eta_1, \ldots, \eta_n)$ is a bijection from the set of morphisms $f : \mathcal{M} \to \mathcal{V}$ to the set of systems of m even functions a_i and n odd functions β_j on $\mathcal{M}$, such that the values $(a_1(x), \ldots, a_m(x))$ are in $V \subset \mathbb{R}^m$ for any $x \in M$.*

For a full proof, see [Lei80], [Man97], [Var04]. The key point of the proof consists in giving a natural meaning to the pullback of arbitrary functions F from $\mathcal{V} \subseteq \overline{\mathbb{R}^{m|n}}$ to $\mathcal{M}$. Let $(t_1, \ldots, t_m, \xi_1, \ldots, \xi_n)$ be the coordinates of $\mathcal{V}$. Assume that $\mathcal{M}$ is a domain $\mathcal{U} \subseteq \overline{\mathbb{R}^{p|q}}$ with coordinates $x_1, \ldots, x_p$ and $\theta_1, \ldots, \theta_q$. Then any function $f \in \mathcal{O}_{\mathcal{M}}(M)$ can be written as

$$f = \sum_I f_I \theta_I. \tag{2.49}$$

Here $I \subset \{1, \ldots, q\}$ is a multiindex running through all increasingly ordered subsets, θ_I is the product of the appropriate odd coordinates in the same order and the f_I are ordinary smooth functions on U. Denote the morphism as $\Phi = (\phi, \varphi) : \mathcal{M} \to \mathcal{V}$. Then the pullbacks $a_i = \varphi(t_i)$, $1 \leq i \leq m$, of the even coordinates of $\mathcal{V}$ are even sections of $\mathcal{O}_{\mathcal{M}}$, thus they are given by elements of the form $a_{i0} + r_i$, where a_{i0} is just a smooth function on V and r_i is even and nilpotent. Now let $F(t_1, \ldots, t_m)$ be a purely even smooth function on V. One then writes its pullback as the formal Taylor series

$$F(a_1, \ldots, a_m) = \sum_{\mathbf{k} \subseteq \{1, \ldots, m\}} \partial^{\mathbf{k}} F(a_{10}, \ldots, a_{m0}) \frac{r_{\mathbf{k}}}{|k|!}. \tag{2.50}$$

The sum runs over all multiindices taking values in $\{1, \ldots, m\}$, $|k|$ is the length of the multiindex, and $r_{\mathbf{k}}$ is the product of the corresponding r_{k_i}'s. Since the r_i's are nilpotent, the sum terminates after finitely many summands.

The pullback of a general function $F = \sum_I F_I \xi_I$ on $\mathcal{V}$ is now uniquely fixed by (2.50) and the requirement that φ be a homomorphism. It can be written as

$$F(a_1, \ldots, a_m, \beta_1, \ldots, \beta_n) = \sum_I F_I(a_1, \ldots, a_m) \beta_I, \tag{2.51}$$

with $\beta_j = \varphi(\xi_j)$.

Thus, prescribing n odd and m even functions whose underlying maps form a continuous map $M \to V$ fixes the morphism Φ uniquely, if one adopts the above prescription (2.50). But why should one choose this definition? What other choices are there? In the next sections it will be shown that there is actually no other possibility to define coordinates. The Taylor-like formula (2.50) is forced upon us by the algebraic properties of supercommutative algebras.

2.2.5 Splitness

One obvious way to construct a supermanifold of dimension $m|n$ is to take an ordinary manifold M of dimension m and a smooth (resp. holomorphic, whichever category M is in and $\mathcal{M}$ should be in) vector bundle $E \to M$ of rank n. Then simply set $\mathcal{M} = (M, \wedge^\bullet E)$, i.e. the structure sheaf to be just the set of sections of the exterior bundle over E, and parity to be given by the degree in the exterior algebra mod 2. This amounts to declaring the fibers of E to be purely odd super vector spaces. The transition functions of such a supermanifold are just smooth transition functions on the base manifold and linear maps in the fibers of E, which then induce the transition functions of $\wedge^\bullet E$. It is clear that not every supermanifold is of this form, since one may in general have maps like

$$\theta'_i = A^i_j(x)\theta_j + A^i_{jkl}(x)\theta_j\theta_k\theta_l + \ldots \tag{2.52}$$

between the odd coordinates of overlapping patches, where A^i_j and A^i_{jkl} are independent collections of smooth (resp. holomorphic) functions. So, while $\wedge^\bullet E$ is

always naturally $\mathbb{Z}$-graded, the structure sheaf of a supermanifold $\mathcal{M}$ is not. However, it is always filtered, namely by the powers of the nilpotent ideal $\mathcal{J} \subset \mathcal{O}_{\mathcal{M}}$:

$$\mathcal{O}_{\mathcal{M}} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \mathcal{J}^3 \supset \dots \tag{2.53}$$

One can reconstruct a vector bundle over the underlying manifold from this filtration. If we set $\mathcal{O}_M = \mathcal{O}_{\mathcal{M}}/\mathcal{J}$ and $E = \Pi(\mathcal{J}/\mathcal{J}^2)$, then $\mathcal{O}_M$ is the structure sheaf of the underlying manifold. Clearly, E can be given the structure of a locally free module of rank n over $\mathcal{O}_M$, where n is the odd dimension of the superdomain. Namely, if f is a section of $\mathcal{O}_{\mathcal{M}}$ and v is a section of $\mathcal{J}$ then the module structure is just given by $[f][v] = [fv]$ where the square brackets denote equivalence classes with respect to the respective quotienting. This way we assign a vector bundle $E \to M$ to $\mathcal{M}$. One checks that this construction is functorial in $\mathcal{M}$. Therefore we call this vector bundle the *canonical vector bundle* associated with $\mathcal{M}$. But as in the case of the reduced manifold we do not get a canonical morphism $\mathcal{M} \to (M, \wedge^\bullet E)$.

Definition 2.2.9. *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold (smooth or complex), and let $E = \Pi(\mathcal{J}/\mathcal{J}^2)$ be the vector bundle over M associated to $\mathcal{M}$. Then $\mathcal{M}$ is called split, if it is isomorphic as a supermanifold to $(M, \wedge^\bullet E)$.*

Lemma 2.2.10. *Every supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ gives rise to a canonical vector bundle $\mathcal{M}^{split} = (M, \wedge^\bullet(\mathcal{O}_{\mathcal{M}}/\mathcal{J}^2))$ and a canonical morphism $P^{split}_{\mathcal{M}} : \mathcal{M}^{split} \to \mathcal{M}$.*

Proof. It is clear from the above discussion that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ always gives rise to a locally free module $E = \Pi(\mathcal{J}/\mathcal{J}^2)$ over the structure sheaf $\mathcal{O}_M$ of the base. Thus, we can always construct a split supermanifold $(M, \wedge^\bullet E)$ from $\mathcal{M}$. Besides, there is always a canonical epimorphism

$$q : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}/\mathcal{N}^2_{\mathcal{M}}. \tag{2.54}$$

Therefore, one always has a morphism of supermanifolds $(\mathrm{id}_M, q \circ \iota) : \mathcal{M}^{split} \to \mathcal{M}$, where $\iota : \mathcal{O}_M \oplus E \hookrightarrow \wedge^\bullet E$ is the canonical inclusion. $\square$

The obstructions to splitness were characterized in terms of sheaf cohomology by Rothstein in [Rot85]. It turns out that in the smooth case, there are no obstructions: every smooth supermanifold is isomorphic to the exterior bundle of a smooth vector bundle. That follows from the fact that in the smooth category all involved sheaves are fine because they are modules over the sheaf of smooth functions on the base. This fact is also known as Batchelor's theorem [Bat79]. In the complex case, that does not hold anymore. In this work, we will be concerned only with complex $1|1$-dimensional manifolds, so splitness plays no role for them. But what will play a role is the fact that also a split supermanifold is not necessarily presented in the form $(M, \wedge^\bullet E)$, its transition functions may still involve higher nilpotent corrections. In particular, if we allow arbitrary coordinate changes,

then we can never assure that a split supermanifold stays in its exterior bundle form. The sets of morphisms between supermanifolds, split or not, are much larger than that of the associated exterior bundles. Since we want to divide out the action of the superdiffeomorphism group, this makes it impossible to simply exploit splitness and consider all the structures we are interested in just in the case of supermanifolds of the form $(M, \wedge^\bullet E)$.

2.2.6 Explicit description of morphisms of supermanifolds

The fact that every supermanifold can be "cut down" to a vector bundle turns out to be also very useful for the description of morphisms. In fact, every morphism between supermanifolds can be seen to be a morphism of vector bundles composed with higher order nilpotent corrections. These corrections themselves form a unipotent group whose construction will be sketched below. We will not try to prove every detail here because much of this has already been done elsewhere (albeit in a somewhat different form, see, e.g., [Leiar], [Var04]). Besides, for the 2|2-dimensional case for which we want to employ these results later, things become much simpler and we will work them out specifically then.

Consider a morphism $\mathcal{M} \to \mathcal{N}$ of supermanifolds. By locality, it is sufficient to look at the case where $\mathcal{M} = \mathcal{U}$ and $\mathcal{N} = \mathcal{V}$ are superdomains. Let $(x_1, \ldots, x_m, \theta_1, \ldots, \theta_n)$ be coordinates on $\mathcal{U}$ and $(y_1, \ldots, y_p, \eta_1, \ldots, \eta_q)$ be coordinates on $\mathcal{V}$. The morphism consists of a continuous map $\phi : U \to V$ of the underlying topological spaces, and of a sheaf map $\varphi : \phi^*\mathcal{O}_\mathcal{V} \to \mathcal{O}_\mathcal{U}$, where $\phi^*\mathcal{O}_\mathcal{V}$ is the pullback of $\mathcal{V}$. Let us denote the associated vector bundles of $\mathcal{U}$ and $\mathcal{V}$ as E and F, respectively. Denote the images of the coordinates of $\mathcal{V}$ as

$$\varphi(y_i) = a_0^i(x) + a_{jk}^i(x)\theta_i\theta_j + \ldots + a_I^i(x)\theta_I + \ldots \tag{2.55}$$

$$\varphi(\eta_j) = b_k^j(x)\theta_k + b_{klm}^j(x)\theta_k\theta_l\theta_m + \ldots + b_I^j(x)\theta_I + \ldots . \tag{2.56}$$

Now if we reduce to $\mathcal{U}^{split}$ by composing Φ with $P_\mathcal{U}^{split}$, then only superfunctions on $\mathcal{U}$ which are linear in the θ_i's are allowed. This implies that φ reduces to a morphism of the associated split manifolds $(U, \wedge^\bullet E) \to (V, \wedge^\bullet F)$, because we are left with

$$\begin{aligned} \varphi(y_i) &= a_0^i(x) \\ \varphi(\eta_j) &= b_k^j(x)\theta_k. \end{aligned} \tag{2.57}$$

It is a classical consequence of the Weierstraß approximation theorem that a smooth morphism between smooth domains is uniquely determined by the images of the coordinate functions under the pullback. Therefore, the sections $a_0^i(x) \in C_U^\infty(U)$, $1 \leq i \leq p$ determine a smooth morphism $\phi : (U, C_U^\infty) \to (V, C_V^\infty)$. The smooth functions b_k^j, $1 \leq k \leq n$, $1 \leq j \leq q$ on the other hand, determine a morphism of vector bundles $\phi^*F \to E$. We will call (2.57) the *associated morphism of canonical bundles* of φ, denoted as $\varphi^{split} : \mathcal{U}^{split} \to \mathcal{V}^{split}$.

Lemma 2.2.11. *Let $\mathcal{M}, \mathcal{N}$ be smooth supermanifolds and let $E \to M$, $F \to N$ be the respective canonical vector bundles. Then every morphism $\Phi : \mathcal{M} \to \mathcal{N}$ induces an associated morphism of exterior bundles $\Phi^{split} : (M, \wedge^\bullet E) \to (N, \wedge^\bullet F)$.*

Proof. Set $\Phi^{split} = P^{split}_{\mathcal{M}} \circ \Phi$. This induces a morphism which has locally the form (2.57), i.e. a smooth bundle morphism $(M, E) \to (N, F)$. □

Now let us start to add the nilpotent corrections to the bundle map (2.57). Let us suppose we only divide out $\mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}/\mathcal{J}^3_{\mathcal{U}}$. Then we have transition functions

$$\begin{aligned} \varphi(y_i) &= a^i_0(x) + a^i_{jk}(x)\theta_i\theta_j \\ \varphi(\eta_j) &= b^j_k(x)\theta_k. \end{aligned} \tag{2.58}$$

Since φ is a homomorphism, we must have

$$\begin{aligned} \varphi(y_i y_j) &= \varphi(y_i)\varphi(y_j) \\ &= a^i_0(x)a^j_0(x) + (a^i_0(x)a^j_{km}(x) + a^i_{km}(x)a^j_0(x))\theta_k\theta_m. \end{aligned} \tag{2.59}$$

Therefore, $a^i_{km}(x)$ is a derivation of $C^\infty_U(U)$, i.e. we can write

$$\varphi(y_i) = \left(1 + \sum_{k,m} a^i_{km}(x)\theta_k\theta_m \frac{\partial}{\partial(a^i_0(x))}\right) a^i_0(x). \tag{2.60}$$

This means, we additionally have to choose p vector fields of degree 2 in the odd variables to determine the morphism completely. Allowing terms of order 3 in the θ_i's will switch on the first correction to $\varphi(\eta_j)$:

$$\varphi(\eta_j) = b^j_k(x)\theta_k + b^j_{klm}(x)\theta_k\theta_l\theta_m. \tag{2.61}$$

Obviously, this can as well be written as

$$\varphi(\eta_j) = \left(1 + \sum_{k,l,m} b^j_{klm}(x)\theta_k\theta_l\theta_m \frac{\partial}{\partial(b^j_k(x)\theta_k)}\right) b^j_k(x)\theta_k. \tag{2.62}$$

So here again, we have to choose q additional vector fields of degree 2 in the odd variables to describe the morphism φ. We can now inductively consider $\mathcal{O}/\mathcal{J}^n$ for $n \to \infty$, and it is clear that in each step $n \to n+1$, we have to choose additional vector fields. Note here the analogy of this construction with the formal Taylor series (2.50). In fact, our construction here justifies the ad-hoc definition (2.50) by showing that there is actually no other way to construct the pullback map.

To write up the result in a readable way, let us denote the images of the coordinates y_i, η_j under the associated bundle morphism φ^{split} (2.57) as

$$a_i := \varphi^{split}(y_i) \qquad \in \mathcal{O}_{\mathcal{U}}/\mathcal{J}_{\mathcal{U}} \tag{2.63}$$

$$\beta_j := \varphi^{split}(\eta_j) \qquad \in \Gamma(E), \tag{2.64}$$

where $E := \Pi(\mathcal{J}_{\mathcal{U}}/\mathcal{J}_{\mathcal{U}}^2)$ is the vector bundle associated with $\mathcal{U}$. Then the image of $\mathcal{O}_{\mathcal{V}}$ under the sheaf map φ^{split} comes endowed with an action of the derivations

$$\frac{\partial}{\partial a_i}, \frac{\partial}{\partial \beta_j}. \tag{2.65}$$

We can generate an algebra of even vector fields from the derivations (2.65) over the sheaf $\mathcal{O}_{\mathcal{U}}$, which we will denote as $\mathcal{T}_{\mathcal{V}}\mathcal{U}$. It is, of course, again filtered by the powers of $\mathcal{J}_{\mathcal{U}}$. Let us write $\mathcal{T}_{\mathcal{V}}\mathcal{U}^{(j)}$ for those elements of $\mathcal{T}_{\mathcal{V}}\mathcal{U}$ whose coefficient function is of order $\geq j$ in the odd variables θ_i. E.g., elements of $\mathcal{T}_{\mathcal{V}}\mathcal{U}^{(2)}$ are given by vector fields of the form

$$f_{ij}^k(x)\theta_i\theta_j\frac{\partial}{\partial a_k}, \quad g_{ijm}^l(x)\theta_i\theta_j\theta_m\frac{\partial}{\partial \beta_l}, \quad h_{ijkl}^n(x)\theta_i\theta_j\theta_k\theta_l\frac{\partial}{\partial a_n}, \ldots.$$

Since the a_k, β_l are not independent of the x_m, θ_n, the Lie algebra $\mathcal{T}_{\mathcal{V}}\mathcal{U}$ is not abelian, although it might look like that at first.

Lemma 2.2.12. *The vector fields $\mathcal{T}_{\mathcal{V}}\mathcal{U}^{(2)}$ generate a nilpotent group $N_{\mathcal{V}}$ under the exponential mapping:*

$$\exp : \mathcal{T}_{\mathcal{V}}\mathcal{U}(2) \to N_{\mathcal{V}}. \tag{2.66}$$

This group acts naturally on $\varphi^{split}(\mathcal{O}_{\mathcal{V}})$:

$$N_{\mathcal{V}} : \varphi^{split}(\mathcal{O}_{\mathcal{V}}) \to \mathcal{O}_{\mathcal{U}}. \tag{2.67}$$

The exponential mapping is well defined because of the nilpotency of the coefficient functions of vector fields in $\mathcal{T}_{\mathcal{V}}\mathcal{U}$, and the inverse of $\exp(X)$ is $\exp(-X)$, as one easily checks. To illustrate the action of $N_{\mathcal{V}}$ on $\varphi^{split}(\mathcal{O}_{\mathcal{V}})$, consider a purely even function $F(y_1, \ldots, y_p)$ on $\mathcal{V}$. Its image under φ^{split} is $F(a_1, \ldots, a_p)$. Let then $X = h(x)\theta_1\theta_2\frac{\partial}{\partial a_1}$ be an element of $\mathcal{T}_{\mathcal{V}}\mathcal{U}^{(2)}$. We have

$$\exp(X) = 1 + h(x)\theta_1\theta_2\frac{\partial}{\partial a_1} \tag{2.68}$$

and thus

$$\exp(X)(F) = F + h(x)\theta_1\theta_2\frac{\partial F}{\partial a_1}. \tag{2.69}$$

Then the main result is the following.

Theorem 2.2.13. *Let $\mathcal{U}$, $\mathcal{V}$ be two superdomains with coordinates $(x_1, \ldots, x_m, \theta_1, \ldots, \theta_n)$ and $(y_1, \ldots, y_p, \eta_1, \ldots, \eta_q)$, respectively. Let $(U, \wedge^\bullet E)$ and $(V, \wedge^\bullet F)$ be the associated split supermanifolds. Let $\Phi = (\phi, \varphi) : \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains, and let φ^{split} be the associated morphism of canonical bundles. Then one can factorize φ into φ^{split} and an element of the nilpotent group $N_{\mathcal{V}}$ constructed in Lemma (2.2.12):*

$$\varphi = \exp(X) \circ \varphi^{split} \tag{2.70}$$

where $X \in \mathcal{T}_{\mathcal{V}}\mathcal{U}^{(2)}$ is an element of the Lie algebra of $N_{\mathcal{V}}$.

This theorem has a couple of very useful consequences. It allows one to control the deviation from the exterior bundle form produced by coordinate changes on a split supermanifold in cases where one cannot retain the form $(M, \wedge^\bullet E)$. Besides, it gives information about the behaviour of geometric objects, e.g., tensor fields, on the supermanifold under morphisms. When one starts with a supermanifold of the form $\mathcal{M} = (M, \wedge^\bullet E)$, then all vector and tensor fields on $\mathcal{M}$ have a natural interpretation in terms of the bundle E. Applying a transformation which involves elements of the nilpotent group $N_{\mathcal{M}}$, this interpretation no longer holds because terms are appearing which are proportional to products of the odd variables. But these additional terms, as one can see in (2.69), are all infinitesimal: they are proportional to Lie derivatives. In the case where the morphisms Φ are superdiffeomorphisms, which will be studied in chapter 7, the analogue of Thm. 2.2.13 takes on a more lucid and practical form: it splits $\widehat{\mathcal{SD}}(\mathcal{M})$ into automorphisms of bundles and a subgroup of nilpotent corrections.

2.2.7 Products

The existence of direct products in the category FinSMan is also assured by Thm. 2.2.8.

Definition 2.2.14. *Let $\mathcal{U} \subseteq \overline{\mathbb{R}^{m|n}}, \mathcal{V} \subseteq \overline{\mathbb{R}^{p|q}}$ be two superdomains with underlying domains U, V. Then $\mathcal{U} \times \mathcal{V}$ is the superdomain in $\overline{\mathbb{R}^{m+p|n+q}}$ whose underlying domain is $U \times V$.*

Thm. 2.2.8 tells us that the homologically defining property

$$\mathrm{Hom}(\mathcal{W}, \mathcal{U} \times \mathcal{V}) \cong \mathrm{Hom}(\mathcal{W}, \mathcal{U}) \times \mathrm{Hom}(\mathcal{W}, \mathcal{V}) \tag{2.71}$$

holds for every superdomain $\mathcal{W}$.

The generalization to supermanifolds is then performed as usually by taking products of domains [Leiar]. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}}), \mathcal{N} = (N, \mathcal{O}_{\mathcal{N}})$ be two supermanifolds. Then the topology of M has a base $B_M = \{U_i\}_{i \in I}$ such that $\mathcal{M}\big|_{U_i}$ is isomorphic to a superdomain for each $i \in I$. Likewise, N has such a base B_N. Choosing isomorphisms $u_i : \mathcal{M}\big|_{U_i} \to \mathcal{U}_i$ and $v_j : \mathcal{N}\big|_{V_j} \to \mathcal{V}_j$ which cover $\mathcal{M}$ and $\mathcal{N}$ with superdomains and appropriate isomorphisms u_{ik}, v_{jl}, we can construct a supermanifold by gluing the superdomains $\mathcal{U}_i \times \mathcal{V}_j$ for all $(i, j) \in I \times J$ via the isomorphisms (u_{ik}, v_{jl}) for all $(i, j), (k, l) \in I \times J$. This manifold is defined to be the product manifold $\mathcal{M} \times \mathcal{N}$. For more details, see [Leiar].

2.2.8 The category of superpoints

A particular class of finite-dimensional supermanifolds are the superpoints $\mathcal{P}(\Lambda_n)$.

Definition 2.2.15. *A finite-dimensional supermanifold whose underlying manifold is a one-point topological space is called a superpoint.*

Comparison with definition 2.2.2 shows that superpoints are the supermanifolds associated to purely odd super vector spaces, i.e., to those of dimension $0|n$. Together with their morphisms as supermanifolds, finite-dimensional superpoints form a category SPoint which is a full subcategory of SMan.

Proposition 2.2.16 (see also [Mol84])**.** *There exists an equivalence of categories*

$$\mathcal{P} : \mathsf{Gr}^\circ \to \mathsf{SPoint}. \tag{2.72}$$

Proof. Define the functor $\mathcal{P}$ on the objects Λ_n of Gr° as

$$\mathcal{P}(\Lambda_n) := \mathcal{P}_n := (\{*\}, \Lambda_n). \tag{2.73}$$

To every morphism $\varphi : \Lambda_n \to \Lambda_m$ of Grassmann algebras, assign the morphism

$$\Phi = (\mathrm{id}_{\{*\}}, \varphi) : \mathcal{P}_m \to \mathcal{P}_n. \tag{2.74}$$

To see that this functor establishes an equivalence, note first that it is fully faithful: on the set of morphisms it is a bijection from $\mathrm{Hom}_{\mathsf{Gr}}(\Lambda_n, \Lambda_m)$ to $\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{P}_m, \mathcal{P}_n)$. The last property to check is essential surjectivity, i.e. every superpoint has to be isomorphic to one of the $\mathcal{P}_n$. This is clear from the fact that since a superpoint is a supermanifold its structure sheaf must be a free superalgebra with n odd generators. Since each such algebra is isomorphic to Λ_n, the assertion is proved. □

This equivalence enables one to use just the full subcategory $\{\mathcal{P}_n | n \in \mathbb{N}_0\}$ given as the image of the functor $\mathcal{P}$ instead of the whole category SPoint. Thus, in the following, SPoint denotes just the image of Gr° under the functor $\mathcal{P}$ constructed in Prop. 2.2.16. Its objects will be denoted $\mathcal{P}_n$. One could also have defined $\mathcal{P}$ as a contravariant functor $\mathsf{Gr} \to \mathsf{SPoint}$. Equivalences of this type are also called *dualities of categories.*

2.3 Super vector bundles

As in ordinary geometry, a smooth super vector bundle over a supermanifold $\mathcal{M}$ can be viewed in two ways [DM99]:

1. as a fiber bundle $\pi : \mathcal{V} \to \mathcal{M}$ whose typical fiber is isomorphic to $\overline{\mathbb{R}^{p|q}}$ and whose structure group is a subsupergroup of $GL(p|q)$
2. as a sheaf $\mathcal{V}$ of $\mathcal{O}_{\mathcal{M}}$-modules which is locally free of rank $p|q$.

Even if the fiber has dimension $p|0$, it is not advisable to think of the bundle as being "purely even". Its sheaf of sections is locally isomorphic to $\mathcal{O}^p_{\mathcal{M}}$, and thus contains odd elements as well. For this reason, if $\mathcal{M}$ is not purely even, the grading of the fibres of $\mathcal{V}$ does not decompose the bundle into a sum of an even and an odd bundle.

Definition 2.3.1. *A section of a super vector bundle $\pi : \mathcal{V} \to \mathcal{M}$ is a morphism $\sigma : \mathcal{M} \to \mathcal{V}$ of smooth supermanifolds such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$.*

One should always keep in mind that the fibers of a super vector bundle are, correctly speaking, not super vector spaces. They are linear supermanifolds. Since there is a canonical relation between linear supermanifolds and super vector spaces this does not always make much of a difference. But there are occasions when it plays a role. If $\pi : \mathcal{V} \to \mathcal{M}$ is a super vector bundle whose fibre is the linear supermanifold $\overline{V} = \overline{\mathbb{R}^{1|2}}$, then one can also define the bundle whose fiber is $\overline{\Pi(V)} = \overline{\mathbb{R}^{2|1}}$. The underlying topological spaces of these two bundles are not the same: the first one is a fiber space whose fibers have dimension 1 while the fibers of the second one have dimension two.

2.3.1 The tangent and cotangent bundle of a supermanifold

Let $\mathcal{M}$ be a smooth supermanifold of dimension $p|q$ with structure sheaf $\mathcal{O}_{\mathcal{M}}$. Its tangent sheaf $\mathcal{T}\mathcal{M}$ is, as in ordinary geometry, the sheaf of $\mathbb{R}$-linear derivations of the structure sheaf, i.e., its stalk at $x \in M$ is $\underline{\mathrm{der}}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M},x}, \mathcal{O}_{\mathcal{M},x})$. Let $(t_1, \ldots, t_p, \theta_1, \ldots, \theta_q)$ be local coordinates on $\mathcal{M}$. Then one has the coordinate vectors $\partial/\partial t_i, \partial/\partial\theta_j$, which are defined to act on a smooth function $f = \sum_I f_I \theta_I$ as follows:

$$\frac{\partial}{\partial t_i} f := \sum_I \frac{\partial f_I}{\partial t_i} \theta_I. \tag{2.75}$$

Writing

$$f = \sum_{j \notin I} (f_I \theta_I + (-1)^{p(f_{j,I})} \theta_j f_{j,I} \theta_I), \tag{2.76}$$

then

$$\frac{\partial}{\partial \theta_j} f := \sum_I (-1)^{p(f_{j,I})} f_{j,I} \theta_I. \tag{2.77}$$

Theorem 2.3.2. *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold of dimension $p|q$. Let $(t_1, \ldots, t_p, \theta_1, \ldots, \theta_q)$ be local coordinates on $\mathcal{M}$. Then its tangent sheaf $\mathcal{T}\mathcal{M}$ is a locally free module over $\mathcal{O}_{\mathcal{M}}$ with basis $\partial/\partial t_i, \partial/\partial\theta_j$ for $1 \leq i \leq p$ and $1 \leq j \leq q$.*

For a proof see [Lei80]. This theorem actually legitimizes the term "tangent bundle" for supermanifolds. Sections of the tangent bundle are called vector fields.

Definition 2.3.3. *The cotangent sheaf $\mathcal{T}^*\mathcal{M}$ of a supermanifold $\mathcal{M}$ is the sheaf of modules dual to $\mathcal{T}\mathcal{M}$ in the sense of section 2.1.5, i.e., the sheaf $\underline{\mathrm{Hom}}(\mathcal{T}\mathcal{M}, \mathcal{O}_{\mathcal{M}})$. Sections of $\mathcal{T}^*\mathcal{M}$ are called (differential) 1-forms.*

At this point one has to fix the conventions regarding the definition of the duality pairing between vector fields and forms as well as the handling of the cohomological degree of differential forms. Both $\mathcal{T}\mathcal{M}$ and $\mathcal{T}^*\mathcal{M}$ will be treated as

left modules. We will always denote the pairing between 1-forms and vector fields as a map

$$\langle \cdot, \cdot \rangle : \mathcal{TM} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{TM}^* \to \mathcal{O}_\mathcal{M}. \tag{2.78}$$

This implies $\langle fX, g\omega \rangle = (-1)^{p(g)p(X)} \langle X, \omega \rangle$ for any vector field X, 1-form ω and functions f, g.

The differential is defined as the unique derivation $d : \mathcal{O}_\mathcal{M} \to \mathcal{T}^*\mathcal{M}$ for which

$$\langle X, df \rangle = X(f) \tag{2.79}$$

holds, with X any vector field and f any function. The local dual basis for $\mathcal{T}^*\mathcal{M}$ is denoted by $dt_1, \dots, dt_p, d\theta_1, \dots, d\theta_q$.

Definition 2.3.4. *The algebra of differential forms on a smooth supermanifold $\mathcal{M}$ is defined to be*

$$\Omega^\bullet_\mathcal{M} = \wedge^\bullet \mathcal{T}^*\mathcal{M}. \tag{2.80}$$

Thus the algebra of differential forms is formally defined as usual. The crucial difference is that $d\theta_i \wedge d\theta_j = d\theta_j \wedge d\theta_i$ (cf. definition 2.19). Therefore, if the odd dimension of $\mathcal{M}$ is not zero, $\Omega^n_\mathcal{M} \neq 0$ for all $n \geq 0$. There are no top-degree forms. Integration on supermanifolds instead involves sections of the Berezinian line bundle [Leiar], [Var04], [DM99]. As in classical geometry, there exists a unique extension of d to a derivation of square zero on the algebra $(\Omega^\bullet_\mathcal{M}, \wedge)$, acting by

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge (d\eta). \tag{2.81}$$

Many further properties of differential forms carry over to supermanifolds, e.g. the Poincaré lemma holds [DM99].

Chapter 3

The categorical approach

While ringed spaces might seem easier to handle at first and are also more widespread in the mathematical literature about supersymmetry, the categorical approach to supergeometry is, in a way, more fundamental. The ringed space formulation has the disadvantage of hiding certain aspects of supergeometry, in particular the origin of odd parameters appearing in geometric constructions. The approach of working with "functorial points", i.e., morphism sets from other objects of the category to the object in question, is in a sense also closer to the way the physicists usually do their calculations.

3.1 Preliminaries

3.1.1 Notation

The object class of a category C will be denoted as $\mathrm{Ob}(\mathsf{C})$. If A is an object of the category C, we simply write $A \in \mathsf{C}$ (instead of $A \in \mathrm{Ob}(\mathsf{C})$).

The class of all morphisms of a category C will be denoted as $\mathrm{Mor}\,\mathsf{C}$.

Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. We will denote a functor morphism $\eta : F \to G$ as a family of maps

$$\eta = \{\eta_A : F(A) \to G(A) \in \mathrm{Hom}_{\mathsf{D}}(F(A), G(A)) \big| A \in \mathsf{C}\}. \tag{3.1}$$

For the category of covariant functors $\mathsf{C} \to \mathsf{D}$, we write D^{C}.

The category $\mathbf{2}$ is the category with two objects, $\mathrm{Ob}(\mathbf{2}) = \{1, 2\}$, and only identity arrows.

3.1.2 Representable functors, Yoneda embedding

Let C be a (nonempty) category, A be an object of C, and let $F : \mathsf{C} \to \mathsf{Sets}$ be a functor. Let further

$$\eta : \mathrm{Hom}_{\mathsf{C}}(A, -) \to F \tag{3.2}$$

be a functor morphism of the covariant Hom-functor to F.

Theorem 3.1.1. *The functor* $H^* : \mathsf{C}^\circ \to \mathsf{Sets}^{\mathsf{C}}$*, given by*

$$\begin{array}{rcl} A & \mapsto & \mathrm{Hom}(A,-) \\ (f : A \to B) & \mapsto & \left(\begin{array}{rcl} \mathrm{Hom}(B,-) & \to & \mathrm{Hom}(A,-) \\ u & \mapsto & u \circ f \end{array} \right) \end{array}$$

defines a full embedding called the Yoneda embedding.

A proof can be found in many books on category theory or homological algebra, e.g., in [Mac98], [GM02], [Sch70]. Replacing C with C°, one obtains an embedding $H_* : \mathsf{C} \to \mathsf{Sets}^{\mathsf{C}^\circ}$. The sets $T(A) := \mathrm{Hom}(A,T)$ are called the A-points of T. If C possesses a terminal object t, the t-points $\mathrm{Hom}(t,M)$ are simply called the *points* of M. The above statement (usually referred to as the Yoneda lemma) allows one to consider each category C as a full subcategory of $\mathsf{Sets}^{\mathsf{C}^\circ}$. If one finds it more convenient, one may therefore avoid working with objects of C directly, working with their points instead, which are all sets. Thinking of familiar categories like manifolds or vector spaces, this might seem an unnecessary complication, but in algebraically more involved settings like, e.g., schemes, this approach is a standard tool.

Definition 3.1.2. *A functor* $F : \mathsf{C} \to \mathsf{Sets}$ *is called representable if it is isomorphic to some* $\mathrm{Hom}(A,-)$*-functor.* A *is then called the representing object of* F*.*

3.1.3 Generators

Luckily, to check the representability of a given functor $F : \mathsf{C} \to \mathsf{Sets}$, it is in many categories not necessary to look at all point sets $F(A)$ for all A. As an example, consider the one-element set $\{*\}$. For any set X, one has

$$\mathrm{Hom}_{\mathsf{Sets}}(\{*\}, X) \cong X. \tag{3.3}$$

This reflects the obvious fact that a set can be described by knowing all ways to throw a single element into it. Some other categories also furnish a single object with this property. For example, for the category of smooth manifolds, the manifold $\operatorname{Spec}\mathbb{R} = (\{*\}, \mathbb{R})$ consisting of a single point with structure sheaf $\mathbb{R}$ does the job. It will turn out that SMan does not possess this nice property.

Definition 3.1.3. *Let* C *be a category. A set* $\{G_i\}_{i\in I}$ *for some index set* I *is called a set of generators for* C *if for any pair* $f, g : A \to B$ *of distinct morphisms between objects* $A, B \in \mathsf{C}$ *there exists* $i \in I$ *and a morphism* $s : G_i \to A$ *such that the compositions*

$$G_i \xrightarrow{s} A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{3.4}$$

are still distinct.

A set of generators is thus able to keep all distinct morphisms distinct. Obviously, if a generator set exists, then it need not at all be unique. To find a suitable one can be non-trivial. The Hom-sets $\mathrm{Hom}(G_i, X)$ can be used to describe the object X but in general they do not encode the full information about X. For example, knowing the set $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{K}, M)$ for a smooth manifold is not sufficient to reconstruct M. In fact, this set only reproduces the underlying set of M but forgets about the topology and smooth structure. Nonetheless these sets can be very useful if we retain the missing information seperately. In the above example we *can* reproduce M if we keep the inner Hom object $\underline{\mathrm{Hom}}(\mathrm{Spec}\,\mathbb{K}, M)$ instead of just the Hom-set. In fact, this inner Hom object is canonically isomorphic to M as one easily checks from the definition.

Conversely, if we give a set S a topology and a manifold structure we may say we define a manifold by assuming that this set is now an inner Hom object. In the case of ordinary manifolds this is, of course, a trivial statement. But this is due to the fact that there exists a single generator in this case. For more subtle objects which cannot be reduced to a single set with structure this reasoning opens up a way to construct them, namely be specifying all the data needed to describe an inner Hom-object (by it's functor of points) and then to prove that there exists an object which represents it.

The following Proposition is sometimes useful.

Proposition 3.1.4. $\{\mathbb{K}^{1|1}\}$ *is a generator set for* $\mathsf{SVec}_{\mathbb{K}}$.

Proof. First we prove that $\{\mathbb{K}^1\}$ is a generator set for the category of ordinary $\mathbb{K}$-vector spaces. Let V, W be two such vector spaces, and let $f, g : V \to W$ be two distinct linear maps. Since $f \neq g$, there exists $v \in V$, $v \neq 0$, such that $f(v) \neq g(v)$. Then f, g differ on the one-dimensional linear subspace spanned by v in V. Let z denote the basis vector of $\mathbb{K}^1$. Then the linear map $\varphi : \mathbb{K}^1 \to V$ given by $\varphi(z) = v$ separates f and g.

Now since two morphisms of super vector spaces may differ either in an even or an odd subspace, we need one even and one odd dimension to separate all distinct morphisms between super vector spaces. □

The set $\{\mathbb{K}^{1|0}, \mathbb{K}^{0|1}\}$ is evidently a set of generators as well. Often it is more convenient to use this set than $\mathbb{K}^{1|1}$.

3.1.4 Direct products, direct sums and fiber products

As seen in the previous chapter, the category SMan has direct products. There is also a categorical characterization of products [GM02]. Following the representable functor philosophy, a direct product $X \times Y$ is the object Z which represents the functor $\mathsf{C} \to \mathsf{Sets}$ given by

$$U \mapsto \{X(U) \times Y(U)\} \text{ in } \mathsf{Sets}. \tag{3.5}$$

Thus, by definition, the functor of points commutes with direct products.

An alternative equivalent definition states that the direct product of X and Y is the limit [GM02] of the functor

$$\begin{aligned} F : \mathbf{2} &\to \mathsf{C} \\ F(1) = X \quad &, \quad F(2) = Y. \end{aligned} \tag{3.6}$$

That means the product is an object Z with projection morphisms $\pi_X : Z \to X$, $\pi_Y : Z \to Y$ which is universal: for any other object Z' also possessing projections π'_X, π'_Y there exists a unique morphism $h : Z' \to Z$ such that $\pi_X \circ h = \pi'_X$ and $pi_Y \circ h = \pi'_Y$.

The direct sum $X \oplus Y$ is the "co"-version of the direct product. In homological terms, it is the colimit of the functor (3.6). This means it is an object Z with canonical injections $\iota_X : X \to Z$, $\iota_Y : Y \to Z$ which is universal among all objects with these properties.

Analogously, given morphisms $\phi : X \to T$ and $\psi : Y \to T$, one can define the fiber product $X \times_T Y$ as the representing object for the functor

$$U \mapsto \{X(U) \times_T Y(U)\} \text{ in } \mathsf{Sets}, \tag{3.7}$$

or as a universal object with projections $p_X : X \times_T Y \to X$, $p_Y : X \times_T Y \to Y$, such that $\phi \circ p_X = \psi \circ p_Y$. But there is another way to define the fiber product which is quite insightful.

Definition 3.1.5. *Let C be some category and $T \in \mathsf{C}$ be an object. Define the category C/T of objects over T in the following way. An object of C/T is a pair (X, ϕ), where $X \in \mathsf{C}$ is an object and ϕ is an arrow $X \to T$. A morphism $\eta : (X, \phi) \to (Y, \varphi)$ is a morphism $\eta \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ such that*

$$\begin{tikzcd} X \arrow[rr, "\eta"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\varphi"] \\ & T & \end{tikzcd} \tag{3.8}$$

commutes. Composition of morphisms consists in the juxtaposition of commutative triangles.

The category C/T is sometimes also called an arrow category or comma category. It is, in geometric terms, the category of bundles over the base T. Then one can show:

Proposition 3.1.6. *Let C be some category with direct products and let T be an object of C. Let $\phi : X \to T$ and $\psi : Y \to T$ be two objects of C/T. Then the fiber product $X \times_S Y$ is the direct product in C/T, i.e. there exists a canonical isomorphism*

$$(X \times_T Y \text{ in } \mathsf{C}) \cong (\phi \times \psi \text{ in } \mathsf{C}/T). \tag{3.9}$$

Proof. Denote the canonical projections of the fiber product as above as $p_X : X \times_T Y \to X$ and $p_Y : X \times_T Y \to Y$. In the category C/T, the direct product of ϕ and ψ is a morphism $h : Z \to T$ together with morphisms $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, "h"] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\phi"'] & T \end{tikzcd} \quad . \tag{3.10}$$

Therefore we have $\psi \circ \pi_Y = \phi \circ \pi_X$ and universality then entails that there exists a unique isomorphism $\eta : Z \to X \times_T Y$. □

Thus, the fiber product of a pair of morphisms of two objects of C into the same base can be thought of as the direct product in the category of bundles of objects of C over that base.

3.2 Algebraic structures in a category

In this entire section, C denotes a category with direct products and a terminal object t. An algebraic structure in C is an object of C which also "has" this type of structure. For concrete categories, i.e., those whose objects all have a natural description as sets with additional properties, such constructions are well-known. For example, a Lie group is a group which is also a manifold and whose operations are diffeomorphisms, or a topological vector space is a topological space which is also a module over a field and whose operations are continuous. The functor of points enables one to extend this concept far beyond this concrete context. The idea is, however, still intriguingly simple: one defines an object $T \in \mathsf{C}$ to be some algebraic structure, if it induces this structure on all sets of morphisms of other objects of C into it. A group in C, for example, is then an object which induces a group structure on all Hom-sets of other objects to it. In the concrete cases, this coincides with the usual intuition.

3.2.1 Rings, modules, algebras

Definition 3.2.1. *An object $R \in \mathsf{C}$ is called a ring with unit in C, if there exist morphisms*

$$+ : R \times R \to R \tag{3.11}$$
$$\cdot : R \times R \to R \tag{3.12}$$
$$e : p \to R, \tag{3.13}$$

such that for every $Y \in \mathsf{C}$, the sets $R(Y) = \mathrm{Hom}(Y, R)$ together with the morphisms $+_Y : R(Y) \times R(Y) \to R(Y)$ and $\cdot_Y : R(Y) \times R(Y) \to R(Y)$ form a ring

and $e_Y : p(Y) \to R(Y)$ is its unit. The ring is called commutative if all rings $R(Y)$ are commutative.

We have implicitly considered R as a functor $\mathsf{C} \to \mathsf{Sets}^{\mathsf{C}^\circ}$ in this definition, namely as its associated Hom-functor $\mathrm{Hom}(-, R)$. Consequently, $+, \cdot, e$ have been turned into functor morphisms, and $+_Y, \cdot_Y$, etc. denote their components. We are free to do so by the Yoneda lemma, and will continue to switch between an object and its contravariant Hom-functor whenever this is suitable.

Definition 3.2.2. *Let R be a ring in C. An object $V \in \mathsf{C}$ is called a left R-module if there exists a morphism $\rho : R \times V \to V$ such that for all $Y \in \mathsf{C}$, the sets $(V(Y), \rho_Y)$ are left $R(Y)$-modules. Right modules are analogously defined. Let V, V' be R-modules. A morphism $f : V \to V'$ of R-modules is a functor morphism*

$$f = \{f_Y : V(Y) \to V'(Y) \big| Y \in \mathsf{C}\}, \tag{3.14}$$

such that each f_Y is a morphism of $R(Y)$-modules.

Together with such morphisms the right (left) R-modules in C form a category $\mathsf{Mod}_R(\mathsf{C})$ (${}_R\mathsf{Mod}(\mathsf{C})$). Since all the points $V(Y)$ of a module V are abelian groups, this category is additive. Supermodules can be defined completely analogously.

Definition 3.2.3. *An R-module V in C is called an R-supermodule if it possesses a direct sum decomposition*

$$V = V_{\bar{0}} \oplus V_{\bar{1}}. \tag{3.15}$$

$V_{\bar{0}}$ will be called the even submodule, $V_{\bar{1}}$ the odd one. A morphism of R-super modules is a morphism of R-modules which preserves the parity.

Note that we did not define R itself to be a superring. This is possible, but we do not need it here. For supermodules to exist, C must obviously allow finite direct sums, which will be the case for all categories that will be dealt with in this work. Left (resp. right) R-super modules in C form a category ${}_R\mathsf{SMod}(\mathsf{C})$ (resp. $\mathsf{SMod}_R(\mathsf{C})$). Commutative, resp. supercommutative rings will be sufficient for almost all our purposes. In this case, each left module can be turned into a right module in a natural way (cf. the convention (2.8).

Definition 3.2.4. *Let R be a commutative ring in C and let A be an R-module. If there exists an R-bilinear morphism $\mu : A \times A \to A$ such that for each $Y \in \mathsf{C}$ the pair $(A(Y), \mu_Y)$ is an $R(Y)$-algebra, the pair (A, μ) is called an R-algebra. The algebra is called (anti-)commutative, Lie, or associative if each of the algebras $A(Y)$ has the corresponding property.*

Modules over R-algebras and morphisms of R-algebras are defined in the obvious way.

3.2.2 Multilinear morphisms

Definition 3.2.5. *Let R be a ring in C and let $V_1, \ldots, V_n, V$ be R-modules. Let Z be some object of C. A morphism $f : Z \times V_1 \times \ldots \times V_n \to V$ is called a Z-family of R-n-linear morphisms if for every $Y \in \mathsf{C}$, the map*

$$f_Y : Z(Y) \times V_1(Y) \times \ldots \times V_n(Y) \to V(Y) \tag{3.16}$$

is a $Z(Y)$-family of $R(Y)$-n-linear maps, i.e., for every $z \in Z(Y)$, $f_Y(z, \ldots)$ is $R(Y)$-n-linear.

The set of Z-families of R-n-linear morphisms from $V_1 \times \ldots \times V_n$ to V will be denoted by $L^n_R(Z; V_1, \ldots, V_n; V)$. It inherits the structure of an abelian group from $V(Y)$:

$$(f + f')_Y = f_Y + f'_Y. \tag{3.17}$$

Besides this, $L^n_R(Z; V_1, \ldots, V_n; V)$ can be given a natural $R(Z)$-module structure. If $f : Z \times V_1 \times \ldots \times V_n \to V$ is an R-n-linear morphism and $r : Z \to R$ is an element of $R(Z)$ one sets

$$(rf)_Y(z, v_1, \ldots, v_n) = r_Y(z) f_Y(z, v_1, \ldots, v_n) \tag{3.18}$$

for $z \in Z(Y)$ and $v_i \in V_i(Y)$. This allows us to consider $L^n_R(Z; -; -)$ as a functor $(\mathsf{Mod}_R(\mathsf{C})^\circ)^n \times \mathsf{Mod}_R(\mathsf{C}) \to \mathsf{Mod}_{R(Z)}(\mathsf{Sets})$.

A family of multilinear morphisms over the terminal object p is just called an R-n-linear morphism. The abelian groups of such morphisms are denoted simply by $L^n_R(V_1, \ldots, V_n; V)$.

For supermodules things are quite similar. One can show [Mol84], [Joh02] that the functor $L^n_R(Z; -; -)$ defined above commutes with finite direct sums. This allows one to equip the set of R-n-linear maps between R-supermodules with the structure of a $R(Z)$-supermodule by decomposing it as

$$(L^n_R)_{\bar{i}}(Z; V_1, \ldots, V_n; V) = \bigoplus_{\bar{j} + \sum \bar{i}_k = \bar{i}} L^n_R(Z; V_{1,\bar{i}_1}, \ldots, V_{n,\bar{i}_n}; V_{\bar{j}}). \tag{3.19}$$

The functor $L^n_R(Z; -; -)$ thus becomes a functor into the category of $R(Z)$-super modules.

3.2.3 Internal Hom-functors and tensor products

In many of the most interesting cases, the sets of multilinear maps can be internalized, i.e. turned into modules themselves. Clearly, these will be the inner Hom-objects for the category $\mathsf{Mod}_R(\mathsf{C})$. One has

$$L^n_R(V_1, \ldots, V_n; V) = \mathrm{Hom}_{\mathsf{Mod}_R(\mathsf{C})}(V_1 \times \ldots \times V_n; V). \tag{3.20}$$

Therefore, an inner Hom-object $\mathcal{L}^n_R(V_1,\ldots,V_n;V)$ has to satisfy

$$\begin{array}{ccc} L^1_R(W;\mathcal{L}^n_R(V_1,\ldots,V_n;V)) & & L^{n+1}_R(W,V_1,\ldots,V_n;V) \\ \| & & \| \\ \mathrm{Hom}(W,\mathcal{L}^n_R(V_1,\ldots,V_n;V)) & \cong & \mathrm{Hom}(W\times V_1\times\ldots\times V_n;V) \end{array} \tag{3.21}$$

for all R-modules W, and these isomorphisms must be functorial in W. In different terms, $\mathcal{L}^n_R(V_1,\ldots,V_n;V)$ has to be a corepresenting object for the functor

$$\begin{aligned} L^{n+1}_R(-,V_1,\ldots,V_n;V):\mathsf{Mod}_R(\mathsf{C})^\circ &\to \mathsf{Sets} \\ W &\mapsto L^{n+1}_R(W,V_1,\ldots,V_n;V). \end{aligned} \tag{3.22}$$

If these inner Hom-objects do all exist, one can go further and construct an internal functor

$$\mathcal{L}^n_R:(\mathsf{Mod}_R(\mathsf{C})^\circ)^n\times\mathsf{Mod}_R(\mathsf{C})\to\mathsf{Mod}_R(\mathsf{C}), \tag{3.23}$$

assigning to each sequence of domain modules $V_1,\ldots,V_n$ and each target module V the inner Hom-object of multilinear maps $V_1\times\ldots\times V_n\to V$.

Closely related are tensor products. The category C is said to possess tensor products over a ring $R\in\mathsf{C}$ if for every set $V_1,\ldots,V_n$ of R-modules there exists an R-module $V_1\otimes_R\ldots\otimes_R V_n$ such that the functors $L^1_R(V_1\otimes_R\ldots\otimes_R V_n;-)$ and $L^n_R(V_1,\ldots,V_n;-)$ are isomorphic. That means

$$L^1_R(V_1\otimes_R\ldots\otimes_R V_n;W)\cong L^n_R(V_1,\ldots,V_n;W)\qquad\forall W\in\mathsf{Mod}_R(\mathsf{C}), \tag{3.24}$$

and these sets have to behave functorially under maps $\phi:W\to W'$. But that is just the condition that the functor

$$\begin{aligned} L^n_R(V_1,\ldots,V_n;-):\mathsf{Mod}_R(\mathsf{C}) &\to \mathsf{Sets} \\ W &\mapsto L^n_R(V_1,\ldots,V_n;W) \end{aligned} \tag{3.25}$$

be representable. This definition coincides for ordinary rings and modules with the usual one. In particular, the tensor product is again a universal object: any other module which satisfies (3.24) is canonically isomorphic to $V_1\otimes_R\ldots\otimes_R V_n$.

The existence and properties of internal Hom-functors and tensor products are, in general, more subtle than presented here. One usually has to assure that they are *coherent*, i.e., they behave as their ordinary counterparts in $\mathsf{Mod}_R(\mathsf{Sets})$ with respect to associativity and the unit of $\otimes$ (if there is one). For the categories of importance for the problems in this work we can take coherence for granted by the results of Molotkov [Mol84]. For an in-depth study of such questions, consult [Joh02].

3.3 The functor of points for supermanifolds

3.3.1 Interpretation of the functor of points in terms of families

The sets of points of an object have a nice interpretation which shows that despite their rather abstract definition they are quite close to geometric reasoning.

The category C/T constructed in Definition 3.1.5 has a terminal object: the identity arrow $\mathrm{id}_T : T \to T$. If the category C has finite direct products, which we will always assume, then one has a natural functor $T^* : \mathsf{C} \to \mathsf{C}/T$ which assigns to any object its trivial family over T:

$$T^* : X \mapsto (\pi_T : T \times X \to T). \tag{3.26}$$

To each morphism $\varphi : X \to Y$ it assigns $(\mathrm{id}_T, \varphi) : T \times X \to T \times Y$.

Proposition 3.3.1. *The map* $\Xi : \mathrm{Hom}_{\mathsf{C}}(T, X) \to \mathrm{Hom}_{\mathsf{C}/T}(\mathrm{id}_T, T^*(X))$ *which assigns to each* $\varphi : T \to X$ *the morphism of families*

$$\begin{tikzcd} T \arrow[rr, "(\mathrm{id}_T,\varphi)"] \arrow[dr, "\mathrm{id}_T"'] & & T \times X \arrow[dl, "\pi_T"] \\ & T & \end{tikzcd} \tag{3.27}$$

is a bijection.

A proof can be found, for example, in [Joh02]. In words: one can identify the T-points of X with the ordinary points of $T^*(X)$ and these are nothing other than the global sections of the projection $\pi_T : T \times X \to T$.

If $\mathrm{Hom}(T, X)$ and $\mathrm{Hom}(S, X)$ are the T- and S-points of X, a morphism $u : T \to S$ induces a morphism $\mathrm{Hom}(S, X) \to \mathrm{Hom}(T, X)$ of these points via composition. From the family point of view this morphism corresponds to a base change, i.e., a pullback of the family. To each family $S \times X \to S$ the map $u : T \to S$ associates the family $T \times_S X \to T$ and to every S-point $\phi : S \to X$ it associates $u \circ \phi : T \to X$. Constructions involving the points of some objects only make sense if they are compatible with arbitrary base changes. In this case one calls them *geometric*.

Furthermore, we have the following.

Proposition 3.3.2. *The functor* $T^* : \mathsf{C} \to \mathsf{C}/T$ *preserves direct products, i.e., there is a canonical isomorphism*

$$T^*(X \times Y) \cong T^*(X) \times_T T^*(Y). \tag{3.28}$$

Proof. Denote the direct product of $X, Y \in \mathsf{C}$ by Z. This means we have canonical projections $\pi_X : Z \to X$,$\pi_Y : Z \to Y$, and Z is universal with this property. The

functor T^* translates this situation into the diagram

$$\begin{tikzcd} T \times X \arrow[dr, "\pi_T"'] & T \times Z \arrow[l, "T^*(\pi_X)"'] \arrow[r, "T^*(\pi_Y)"] \arrow[d, "\pi_T"] & T \times Y \arrow[dl, "\pi_T"] \\ & T & \end{tikzcd} . \tag{3.29}$$

It has to be shown that $T \times Z$ together with the morphims $T^*(\pi_X)$ and $T^*(\pi_Y)$ is universal. Let $\eta : Q \to T$ be another object of C/T with morphisms $P_X : Q \to T \times X$, $P_Y : Q \to T \times Y$, such that

$$\begin{tikzcd} T \times X \arrow[dr, "\pi_T"'] & Q \arrow[l, "P_X"'] \arrow[r, "P_Y"] \arrow[d, "\eta"] & T \times Y \arrow[dl, "\pi_T"] \\ & T & \end{tikzcd} \tag{3.30}$$

commutes. Denote by $p_X : T \times X \to X$ and $p_Y : T \times Y \to Y$ the canonical projections. Then we have morphisms $p_X \circ P_X : Q \to X$ and $p_Y \circ P_Y : Q \to Y$, and thus, there exists a unique morphism $\alpha : Q \to Z$ such that $\pi_X \circ \alpha = p_X \circ P_X$ and $\pi_Y \circ \alpha = p_Y \circ P_Y$.

But then we have morphisms $\alpha : Q \to Z$ and $\eta : Q \to T$ and thus there exists a unique morphism $h : Q \to T \times Z$. □

By virtue of Prop. 3.1.6, we can then conclude that T^* maps direct products $X \times Y$ into fiber products $X \times_T Y$. This is important if one has an algebraic structure on an object of the category C, e.g., if some $X \in \mathsf{C}$ is a group in C. Then T^* will produce an object with the same structure in C/T.

3.3.2 Generators for the category of supermanifolds

In order to be able to actually work with the functor of points of a supermanifold we need a manageable set of generators. A single generator is not enough, as it turns out.

Theorem 3.3.3. *The objects of the category* SPoint *(cf. section 2.2.8) form a set of generators for the category* $\mathsf{FinSMan}$ *of finite dimensional super manifolds.*

Proof. The claim is local, therefore it is sufficient to check it for superdomains. Let $\mathcal{U}, \mathcal{V}$ be two superdomains and $f, g : \mathcal{U} \to \mathcal{V}$ be a pair of distinct morphisms. It is well known that $\operatorname{Spec} \mathbb{K} = (\{*\}, \mathbb{K})$ is a generator for the categories of real smooth (resp. complex analytic) finite-dimensional manifolds: these are completely described by their $\mathbb{K}$-points. Thus if the underlying smooth maps of f, g are distinct they will be separated already by $\mathcal{P}_0 = \operatorname{Spec} \mathbb{K}$. Let us therefore assume that the underlying maps of f and g are identical. Since $f \neq g$, there must exist a (topological) point $p \in U$, which gets mapped by f and g to the point $q \in V$ such that the sheaf maps

$$\tilde{f}, \tilde{g} : \mathcal{O}_{\mathcal{V}} \to \mathcal{O}_{\mathcal{U}} \tag{3.31}$$

differ on the stalk at q, and this difference must either be visible in the images of the odd coordinates or in the even nilpotent corrections to the images of the even coordinates. Let x collectively denote the even coordinates of $\mathcal{U}$ and let $\theta_1, \ldots, \theta_n$ be the odd coordinates of $\mathcal{U}$. Let $y_1, \ldots, y_p$ be the even and $\eta_1, \ldots, \eta_r$ be the odd coordinates of $\mathcal{V}$. Let us first assume that $\tilde{f}, \tilde{g}$ differ in the image of some η_i, which, without loss of generality, we assume to be η_1. Then we can write

$$\tilde{f}(\eta_1) = \alpha_i(x)\theta_i + \alpha_{ijk}(x)\theta_i\theta_j\theta_k + \ldots \tag{3.32}$$

$$\tilde{g}(\eta_1) = \beta_i(x)\theta_i + \beta_{ijk}(x)\theta_i\theta_j\theta_k + \ldots \tag{3.33}$$

where α_I, β_I are germs of smooth (resp. holomorphic) functions around $p \in U$. At least for one index J, $\alpha_J \neq \beta_J$ can be assumed. Since these germs are entirely defined by their values, there must be a point p' in any neighbourhood of p where the values of α_J and β_J are different. To separate f and g, we may then define a map $\Phi = (\phi, \varphi) : \mathcal{P}_N \to \mathcal{U}$ by choosing $N \geq n$ and setting $\phi(\{*\}) = p'$. If $\xi_1, \ldots, \xi_N$ are the coordinates of $\mathcal{P}_N$, then the sheaf map is defined to be

$$\begin{aligned} \varphi(\theta_i) &:= \xi_i \\ \varphi(f) &:= f(p) \end{aligned} \tag{3.34}$$

where f is any germ of a function on the underlying manifold U. The image of η_1 under $\tilde{f}, \tilde{g}$ will be mapped to a different element of $\wedge[\xi_1, \ldots, \xi_N]$'s. Thus the two morphisms remain distinct. It is clear that one must be able to choose an arbitrarily large N in order to separate all morphisms between finite-dimensional supermanifolds: to make the difference between α_J and β_J visible one needs at least as many odd coordinates as the length $|J|$ of the index.

If we assume instead that $\tilde{f}, \tilde{g}$ differ on the image of an even coordinate y_k, say y_1, we can write

$$\tilde{f}(y_1) = \alpha_0(x) + \alpha_{ij}(x)\theta_i\theta_j + \ldots \tag{3.35}$$

$$\tilde{g}(y_1) = \beta_0(x) + \beta_{ij}(x)\theta_i\theta_j + \ldots, \tag{3.36}$$

and for some $J \neq 0$, we have $\alpha_J \neq \beta_J$. By the same arguments as before, there exists a p'' in any neighbourhood of p for which the map (3.34) separates $\tilde{f}, \tilde{g}$. $\square$

In general, we will have to keep track of an infinite tower of sets of points, one set for each Λ_n, $n \in \mathbb{N}_0$. This is still much less comfortable than ordinary geometry, but at least allows a clean handling of, e.g., odd parameters or supergroups. It will also provide a means to extend the notion of a supermanifold beyond finite dimensions.

Theorem 3.3.3 also means that a supermanifold can be defined as a contravariant functor $\mathsf{SPoint} \to \mathsf{Sets}$ which is representable, i.e., a family of sets S_n such that there exists a supermanifold $\mathcal{M}$ for which $S_n \cong \mathrm{Hom}(\mathcal{P}_n, \mathcal{M})$ for all $n \in \mathbb{N}_0$. By Prop. 2.2.16, we can as well use covariant functors $\mathsf{Gr} \to \mathsf{Sets}$, and this is precisely the way we will proceed in the next section, following largely the program proposed by Molotkov [Mol84].

3.3.3 Consequences of the family point of view for supermanifolds

Thm. 3.3.3 shows that it is sufficient to consider the $\mathcal{P}_n$-points of any supermanifold $\mathcal{M}$ to describe it entirely. One way to construct a supermanifold is therefore to propose a functor $F : \mathsf{SPoint} \to \mathsf{Sets}$ and show that it is representable. How to do this in detail will be discussed later. At this point we only want to use the family point of view to explain the origin of the odd parameters occuring in coordinate changes on supermanifolds.

The $\mathcal{P}_n$-points of a supermanifold $\mathcal{M}$ are morphisms $\mathcal{P}_n \to \mathcal{M}$. By the interpretation of Prop. 3.3.1 they may be viewed as *local* sections of the bundle $\mathcal{P}_n \times \mathcal{M} \to \mathcal{P}_n$. That means to study a supermanifold one studies families $\phi : \mathcal{M} \to \mathcal{P}_n$ of locally superringed spaces which are locally on $\mathcal{M}$ isomorphic to projections $\mathcal{P}_n \times \mathcal{U} \to \mathcal{P}_n$, where $\mathcal{U}$ is a superdomain. Now suppose one has found two such local sections, $\mathcal{P}_n \times \mathcal{U} \to \mathcal{P}_n$ and $\mathcal{P}_n \times \mathcal{V} \to \mathcal{P}_n$, whose domains $U, V \subset M$ overlap. Then on the overlap one needs an invertible transition function, i.e., an isomorphism ψ such that

$$\begin{tikzcd} \mathcal{P}_n \times \mathcal{U} \arrow[rr, "\psi"'] \arrow[dr, "\pi_{\mathcal{P}_n}"'] & & \mathcal{P}_n \times \mathcal{V} \arrow[dl, "\pi_{\mathcal{P}_n}"] \\ & \mathcal{P}_n & \end{tikzcd} \tag{3.37}$$

commutes. This means that we can redefine ψ as $\pi_{\mathcal{P}_n} \times \psi$ where ψ is now a map $\psi : \mathcal{P}_n \times \mathcal{U} \to \mathcal{V}$. That it is an isomorphism means that there exists an inverse map $\psi^{-1} : \mathcal{P}_n \times \mathcal{V} \to \mathcal{U}$, such that the composition

$$\mathcal{P}_n \times \mathcal{U} \xrightarrow{\pi_{\mathcal{P}_n} \times \psi} \mathcal{P}_n \times \mathcal{V} \xrightarrow{\pi_{\mathcal{P}_n} \times \psi^{-1}} \mathcal{P}_n \times \mathcal{U} \tag{3.38}$$

is the identity $\mathrm{id}_{\mathcal{P}_n \times \mathcal{U}}$. The map ψ may therefore involve the odd generators of Λ_n, i.e., it is stalkwise a map $\mathcal{O}_{\mathcal{U},p} \otimes \Lambda_n \to \mathcal{O}_{\mathcal{V},\psi(p)}$. This "mixing in" of the odd coordinates of the base causes the occurence of the so-called odd parameters in coordinate transformations on a supermanifold.

One may also interpret the occurence of odd parameters in terms of inner Hom-objects. For two supermanifolds $\mathcal{M}$ and $\mathcal{N}$ an inner Hom object $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})$ would have to be a supermanifold of supersmooth maps satisfying

$$\mathrm{Hom}(\mathcal{X}, \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}(\mathcal{X} \times \mathcal{M}, \mathcal{N})$$

for all supermanifolds $\mathcal{X}$. Since we know that the superpoints generate $\mathsf{FinSMan}$, it will be sufficient to verify that one has

$$\mathrm{Hom}(\mathcal{P}_n, \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}(\mathcal{P}_n \times \mathcal{M}, \mathcal{N}) \qquad \forall n \in \mathbb{N}_0. \tag{3.39}$$

Thus, a morphism $\mathcal{P}_n \times \mathcal{M} \to \mathcal{N}$ may be thought of as a "higher point" of a supermanifold of morphisms $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})$, as opposed to the $\mathbb{K}$-points which are just the ordinary morphisms $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})_{red} = \mathrm{Hom}(\mathcal{M}, \mathcal{N})$. This point of

view will be pursued further when the supergroup of superdiffeomorphisms will be studied, which is the restriction of the inner Hom object $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{M})$ to the (ordinary) group $\mathrm{Aut}(\mathcal{M})$. We will also take it up when supermanifolds of sections of vector bundles on a supermanifold are constructed.

3.4 The categorical formulation of linear and commutative superalgebra

As a first step towards a fully categorical formulation of supergeometry, we will translate linear and commutative superalgebra into the language of the functor of points. By the remark following Prop. 3.1.4, the only points that would matter for a linear superspace are the Λ_0- and Λ_1-points. In view of the later extension of the construction to algebras and general supermanifolds, we will nonetheless use a bigger set of generators right away, namely all finite dimensional Grassmann algebras. The construction outlined below can be viewed as a systematic treatment of the so-called even rules principle, which is a way to do superalgebra without having to handle odd quantities [DM99], [Var04].

3.4.1 The rings $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$

All considerations in this and the following sections refer to the category $\mathsf{Sets}^{\mathsf{Gr}}$ of covariant functors $\mathsf{Gr} \to \mathsf{Sets}$.

For the case when we work over the reals, i.e. with the category Gr of real Grassmann algebras, we define in $\mathsf{Sets}^{\mathsf{Gr}}$ a commutative ring $\overline{\mathbb{R}}$ with unity by setting

$$\begin{aligned} \overline{\mathbb{R}}(\Lambda) &:= (\Lambda \otimes_{\mathbb{R}} \mathbb{R})_{\bar{0}} = \Lambda_{\bar{0}}, && (3.40)\\ \overline{\mathbb{R}}(\varphi) &:= \varphi\big|_{\Lambda_{\bar{0}}} && (3.41)\end{aligned}$$

for $\varphi : \Lambda \to \Lambda'$ a morphism in Gr.[1] The ring structure is directly inherited from the one of the $\Lambda_{\bar{0}}$.

If we work over $\mathbb{C}$, i.e., with $\mathsf{Gr}^{\mathbb{C}}$, then we analogously define

$$\begin{aligned} \overline{\mathbb{C}}(\Lambda^{\mathbb{C}}) &:= (\Lambda^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C})_{\bar{0}} = \Lambda^{\mathbb{C}}_{\bar{0}}, && (3.42)\\ \overline{\mathbb{C}}(\varphi) &:= \varphi\big|_{\Lambda^{\mathbb{C}}_{\bar{0}}}. && (3.43)\end{aligned}$$

These two rings will play the role usually played by the ground field in the definition of linear spaces.

[1] To tensor with $\mathbb{R}$ over $\mathbb{R}$ is, of course, not necessary in this definition. We merely used this notation to stress the analogy with the construction of modules in the next section.

3.4.2 Superrepresentable $\overline{\mathbb{K}}$-modules in $\mathsf{Sets}^{\mathsf{Gr}}$

In this section we will introduce a particular class of $\overline{\mathbb{R}}$- and $\overline{\mathbb{C}}$-modules in the functor category $\mathsf{Sets}^{\mathsf{Gr}}$, which, as it will turn out, can be used instead of super vector spaces in the categorical approach.

Let V be some real or complex super vector space. We define a functor $\overline{V} \in \mathsf{Sets}^{\mathsf{Gr}}$ by setting

$$\begin{aligned} \overline{V}(\Lambda) &:= (\Lambda \otimes_{\mathbb{K}} V)_{\bar{0}} = (\Lambda_{\bar{0}} \otimes V_{\bar{0}}) \oplus (\Lambda_{\bar{1}} \otimes V_{\bar{1}}) \\ \overline{V}(\varphi) &:= (\varphi \otimes \mathrm{id}_V)\big|_{\overline{V}(\Lambda)} \qquad \text{for} \quad \varphi : \Lambda \to \Lambda'. \end{aligned} \tag{3.44}$$

Here Λ means the $\mathbb{K}$-version of the Grassmann algebras. All sets $\overline{V}(\Lambda)$ are naturally $\Lambda_{\bar{0}}$-modules, and thus, $\overline{V}$ is a $\overline{\mathbb{K}}$-module.

Let $f : V_1 \times \ldots \times V_n \to V$ be a multilinear map of $\mathbb{K}$-super vector spaces. To f, one assigns the functor morphism $\overline{f} : \overline{V}_1 \times \ldots \times \overline{V}_n \to \overline{V}_n$ whose components

$$\overline{f}_\Lambda : \overline{V}_1(\Lambda) \times \ldots \times \overline{V}_n(\Lambda) \to \overline{V}(\Lambda) \tag{3.45}$$

are defined by

$$\overline{f}_\Lambda(\lambda_1 \otimes v_1, \ldots, \lambda_n \otimes v_n) = \lambda_n \cdots \lambda_1 \otimes f(v_1, \ldots, v_n) \tag{3.46}$$

for all $\lambda_i \otimes v_i \in \overline{V}_i(\Lambda)$. All maps $\overline{f}_\Lambda$ are clearly $\Lambda_{\bar{0}}$-linear, hence $\overline{f}$ is a $\overline{\mathbb{K}}$-n-linear morphism in $\mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}})$.

Proposition 3.4.1 (see also [Mol84])**.** *The assignment $f \mapsto \overline{f}$, which is a map*

$$L^n_{\mathbb{K},\bar{0}}(V_1, \ldots, V_n; V) \to L^n_{\overline{\mathbb{K}}}(\overline{V}_1, \ldots, \overline{V}_n; \overline{V}) \tag{3.47}$$

for any tuple $V_1, \ldots, V_n, V$ of $\mathbb{K}$-super vector spaces, is an isomorphism of $\mathbb{K}$-modules.

Proof. [2] Remember that the $\mathbb{K}$-module structure on $L^n_{\overline{\mathbb{K}}}(\overline{V}_1, \ldots, \overline{V}_n; \overline{V})$ is due to the fact that $L^n_{\overline{\mathbb{K}}}(\overline{V}_1, \ldots, \overline{V}_n; \overline{V})$ is a family of multilinear maps over the final object $\mathbb{K}$ of Gr, and thus, $L^n_{\overline{\mathbb{K}}}(\overline{V}_1, \ldots, \overline{V}_n; \overline{V})$ has the natural structure of a $\overline{\mathbb{K}}(\mathbb{K}) = \mathbb{K}$-module.

Definition (3.46) assigns to every $f \in L^n_{\mathbb{K},\bar{0}}(V_1, \ldots, V_n; V)$ a functor morphism $\overline{f}$. We have to show that one can reconstruct a unique f from a given $\overline{f} \in L^n_{\overline{\mathbb{K}}}(\overline{V}_1, \ldots, \overline{V}_n; \overline{V})$. Let a seqeuence $\lambda_i \otimes v_i \in \overline{V}_i(\Lambda)$ be given, $1 \le i \le n$ and let $j \le n$ of these v_i be odd, assuming for simplicity that these are the first j. By $\Lambda_{\bar{0}}$-linearity we then have

$$\overline{f}_\Lambda(\lambda_1 \otimes v_1, \ldots, \lambda_n \otimes v_n) = \lambda_n \cdots \lambda_{j+1} \overline{f}_\Lambda(\lambda_1 \otimes v_1, \ldots, \lambda_j \otimes v_j, 1 \otimes v_{j+1}, \ldots, 1 \otimes v_n). \tag{3.48}$$

[2] I am grateful to V. Molotkov for pointing out an error in my original proof of this Proposition and for sending me a correct version.

Now consider the morphism $\eta : \Lambda_j \to \Lambda$ which is defined by $\eta(\theta_k) = \lambda_k$, where θ_k, $1 \leq k \leq j$ are the odd generators of Λ_j. In order to prove the statement of the Proposition it will now be enough to show that there exists a unique

$$g \in L^n_{\mathbb{K},\bar{0}}(V_1, \ldots, V_n; V)$$

such that

$$f_{\Lambda_j}(\theta_1 \otimes v_1, \ldots, \theta_j \otimes v_j, 1 \otimes v_{j+1}, \ldots, 1 \otimes v_n) = \theta_i \cdots \theta_1 \otimes g(v_1, \ldots, v_n). \tag{3.49}$$

This is sufficient because the fact that f is a functor morphism implies that we have a commutative square

$$\begin{tikzcd} \overline{V}_1(\Lambda_j) \times \ldots \times \overline{V}_n(\Lambda_j) \arrow[r, "\overline{V}_1(\eta)\times\ldots\times\overline{V}_n(\eta)"] \arrow[d, "\overline{f}_{\Lambda_j}"'] & \overline{V}_1(\Lambda) \times \ldots \times \overline{V}_n(\Lambda) \arrow[d, "\overline{f}_{\Lambda}"] \\ \overline{V}(\Lambda_j) \arrow[r, "\overline{V}(\eta)"] & \overline{V}(\Lambda) \end{tikzcd} \tag{3.50}$$

and this then entails that

$$\overline{f}_{\Lambda}(\lambda_1 \otimes v_1, \ldots, \lambda_n \otimes v_n) = \lambda_n \cdots \lambda_1 \otimes g(v_1, \ldots, v_n)$$

for a unique even linear map g.

To prove that a unique g as claimed in eq. (3.49) exists, we first observe that the most general expression that could appear on the right hand side of eq. (3.49) reads

$$\begin{aligned} f_{\Lambda_j}(\theta_1 \otimes v_1, \ldots, \theta_j \otimes v_j, 1 \otimes v_{j+1}, \ldots, 1 \otimes v_n) = \theta_i \cdots \theta_1 \otimes g(v_1, \ldots, v_n) + \\ + \sum_{\substack{m<j \\ j_1 > \ldots > j_m}} \theta_{j_1} \cdots \theta_{j_m} \otimes g_{j_1 \cdots j_m}(v_1, \ldots, v_n), \end{aligned} \tag{3.51}$$

where all $g_{j_1 \cdots j_m}$ are linear maps. To show that the sum on the right hand side equals zero, we again use functoriality (3.50), this time for the morphism

$$\begin{array}{rcll} \varphi_l : \Lambda_j & \to & \Lambda_j & \\ \varphi_l(\theta_k) & = & 0 & \text{if} \quad k = l \\ \varphi_l(\theta_k) & = & \theta_k & \text{if} \quad k \neq l. \end{array}$$

This evidently kills all summands which contain θ_l and yields

$$0 = \sum_{\substack{m<j \\ l \notin \{j_1, \ldots, j_m\}}} \theta_{j_1} \cdots \theta_{j_m} \otimes g_{j_1 \cdots j_m}(v_1, \ldots, v_n).$$

We obtain such an equation for each $1 \leq l \leq j$, therefore the sum on the right hand side of (3.51) must equal zero. □

This Proposition is of central importance for all further constructions. As a first application, we obtain the following Corollary which shows that $\overline{\mathbb{K}}$-modules in $\mathsf{Sets}^{\mathsf{Gr}}$ are just the right objects to fully capture the structure of a super vector space, i.e., of a $\mathbb{K}$-supermodule in Sets.

Corollary 3.4.2 (see also [Mol84])**.** *The assignment $V \mapsto \overline{V}$ and $f \mapsto \overline{f}$ defines a fully faithful functor*

$$\bar{\cdot} : \mathsf{SMod}_{\mathbb{K}}(\mathsf{Sets}) \to \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}}). \tag{3.52}$$

Proof. It has to be shown that the assignment $f \mapsto \overline{f}$ is a bijection

$$(L_{\mathbb{K}})_{\bar{0}}(V; V') \to L^1_{\overline{\mathbb{K}}}(\overline{V}; \overline{V'}) \tag{3.53}$$

for any pair V, V' of $\mathbb{K}$-super vector spaces. This is a special case of Prop. 3.4.1 if one inserts there $V_1 = V$ and $V = V'$. □

We do not get an equivalence here, because Prop. 3.4.1 only holds for superrepresentable $\overline{\mathbb{K}}$-modules (see Def. 3.4.4). On the other hand we can conclude much more than the above Corollary, since Prop. 3.4.1 does not just make a statement about linear maps between two super vector spaces but about multilinear maps on an arbitrary finite number of arguments.

Corollary 3.4.3. *The functor $\bar{\cdot} : \mathsf{SMod}_{\mathbb{K}}(\mathsf{Sets}) \to \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}})$ induces fully faithful functors*

$$\begin{aligned} \bar{\cdot} : \mathsf{SLie}_{\mathbb{K}}(\mathsf{Sets}) &\to \mathsf{Lie}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}}) && (3.54)\\ \bar{\cdot} : \mathsf{SAlg}_{\mathbb{K}}(\mathsf{Sets}) &\to \mathsf{Alg}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}}) && (3.55) \end{aligned}$$

between the categories of $\mathbb{K}$-super Lie algebras and ordinary $\overline{\mathbb{K}}$-Lie algebras in $\mathsf{Sets}^{\mathsf{Gr}}$, and between $\mathbb{K}$-super algebras and ordinary $\overline{\mathbb{K}}$-algebras.

Proof. A $\mathbb{K}$-super algebra A is a $\mathbb{K}$-super vector space with a bilinear product $\mu : A \otimes A \to A$. Clearly, the functor $\bar{\cdot}$ assigns to A a functor $\overline{A}$ and to μ a $\overline{\mathbb{K}}$-bilinear morphism $\overline{\mu}$ such that $\overline{A}$ becomes a $\overline{\mathbb{K}}$-algebra. By Prop. 3.4.1 we are done if we can show that a morphism $f : (A, \mu) \to (A', \mu')$ of $\mathbb{K}$-super algebras will be mapped to a morphism of the corresponding $\overline{\mathbb{K}}$-algebras. This means we want to verify that for any Λ,

$$\overline{f}_{\Lambda}(\overline{\mu}_{\Lambda}(\lambda_1 \otimes a_1, \lambda_2 \otimes a_2)) = \overline{\mu}'_{\Lambda}(\overline{f}_{\Lambda}(\lambda_1 \otimes a_1), \overline{f}(\lambda_2 \otimes a_2)). \tag{3.56}$$

for all $\lambda_i \otimes a_i \in \overline{A}(\Lambda)$. The left hand side is

$$\overline{f}_{\Lambda}(\lambda_2\lambda_1 \otimes \mu(a_1, a_2)) = \lambda_2\lambda_1 \otimes f(\mu(a_1, a_2)). \tag{3.57}$$

The right hand side can be written as

$$\overline{\mu}'_{\Lambda}(\lambda_1 \otimes f(a_1), \lambda_2 \otimes f(a_2)) = \lambda_2\lambda_1 \otimes \mu'(f(a_1), f(a_2)), \tag{3.58}$$

and that is the same as (3.57) since f was a homomorphism of super algebras. □

One could extend the list of categories between which $\bar{\cdot}$ maps fully faithfully to any kind of $\mathbb{K}$-multilinear superalgebraic structure which is defined by relations involving finitely many arguments, e.g., associative superalgebras [Mol84].

Definition 3.4.4. *A $\overline{\mathbb{K}}$-module $\mathcal{V}$ in $\mathsf{Sets}^{\mathsf{Gr}}$ is called superrepresentable if it is isomorphic to $\overline{V}$ for some $\mathbb{K}$-super vector space V.*

Due to Prop. 3.4.1, the superrepresentable $\overline{\mathbb{K}}$-modules form a full subcategory in $\mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}})$, and $\bar{\cdot}$ is an equivalence between this subcategory and $\mathsf{SMod}_{\mathbb{K}}(\mathsf{Sets})$. Non-superrepresentable $\overline{\mathbb{K}}$-modules do indeed exist. An example that will prove useful later on is $\overline{V}^{nil}$. Let V be some $\mathbb{K}$-super vector space and let Λ^{nil} be the nilpotent ideal in Λ. Then one can define a $\overline{\mathbb{K}}$-module by setting

$$\overline{V}^{nil}(\Lambda) \;:=\; (\Lambda^{nil} \otimes_{\mathbb{K}} V)_{\bar{0}} \tag{3.59}$$

$$\overline{V}(\varphi) \;:=\; (\varphi \otimes \mathrm{id}_V)\big|_{\overline{V}^{nil}(\Lambda)} \qquad \text{for} \quad \varphi : \Lambda \to \Lambda'. \tag{3.60}$$

For every Λ, one has

$$\overline{V}(\Lambda) = V_{\bar{0}} \oplus \overline{V}^{nil}(\Lambda) = V_{\bar{0}} \oplus \left(\Lambda^{nil}_{\bar{0}} \otimes V_{\bar{0}}\right) \oplus \left(\Lambda^{nil}_{\bar{1}} \otimes V_{\bar{1}}\right). \tag{3.61}$$

Hence $\overline{V}^{nil}$ is superrepresentable if and only if $V_{\bar{0}} = 0$, in which case V itself superrepresents it.

Finally, the change of parity functor Π can be carried over to the category of superrepresentable $\overline{\mathbb{K}}$-modules in an obvious way.

Definition 3.4.5. *Let $\mathsf{SRepMod}_{\overline{\mathbb{K}}} \subset \mathsf{SMod}_{\overline{\mathbb{K}}}$ be the full subcategory of superrepresentable $\overline{\mathbb{K}}$-modules. The change of parity functor $\overline{\Pi}$ is defined as*

$$\overline{\Pi} : \mathsf{SRepMod}_{\overline{\mathbb{K}}} \;\to\; \mathsf{SRepMod}_{\overline{\mathbb{K}}} \tag{3.62}$$

$$\overline{V} \;\mapsto\; \overline{(\Pi(V))}. \tag{3.63}$$

3.5 Superdomains

In this section, superdomains will be defined in purely categorical terms as open regions of superrepresentable $\overline{\mathbb{K}}$-modules. Since we can define $\mathbb{K}$-super vector spaces and thus $\overline{\mathbb{K}}$-modules of arbitrary dimensions, infinite-dimensional superdomains will also become available. Later, these will again serve as the building blocks of infinite-dimensional supermanifolds.

To make the notion of an open subobject sensible, one has to define an analogue of a topology on $\overline{\mathbb{K}}$-modules. Since the objects of $\mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{Sets}^{\mathsf{Gr}})$ have no immediate interpretation as sets, this seems somewhat strange at first. The solution is, once more, to use the sets of functorial points, each of which is just an ordinary set and may thus be given a topology in the usual sense. Given an open set in each of these spaces, these open sets will have to behave functorially with respect to Gr in order to form an *open subfunctor* in some $\overline{\mathbb{K}}$-module. The concept one arrives at, pursuing this intuition, is that of a Grothendieck (pre-)topology.

3.5.1 The site $\mathsf{Top}^{\mathsf{Gr}}$

Following the idea outlined above, we study functors $\mathsf{Gr} \to \mathsf{Top}$, i.e. from the finite-dimensional Grassmann algebras into topological spaces.

Definition 3.5.1. *Let $\mathcal{F}, \mathcal{F}'$ be functors in $\mathsf{Top}^{\mathsf{Gr}}$. $\mathcal{F}'$ is called a subfunctor of $\mathcal{F}$, if*

1. *for every $\Lambda \in \mathsf{Gr}$, $\mathcal{F}'(\Lambda)$ is a topological subspace of $\mathcal{F}(\Lambda)$, and*
2. *the family of inclusions $\{\mathcal{F}'(\Lambda) \subset \mathcal{F}(\Lambda) \big| \Lambda \in \mathsf{Gr}\}$ forms a functor morphism.*

In this case, one just writes $\mathcal{F}' \subset \mathcal{F}$. $\mathcal{F}'$ is called an open subfunctor of $\mathcal{F}$ if, in addition, each $\mathcal{F}'(\Lambda)$ is open in $\mathcal{F}(\Lambda)$.

Definition 3.5.2. *Let $\mathcal{F}', \mathcal{F}''$ be open subfunctors of $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$. Then the intersection $\mathcal{F}' \cap \mathcal{F}''$ is the functor whose points are*

$$(\mathcal{F}' \cap \mathcal{F}'')(\Lambda) := \mathcal{F}'(\Lambda) \cap \mathcal{F}''(\Lambda). \tag{3.64}$$

The union $\mathcal{F}' \cup \mathcal{F}''$ is the functor defined by

$$(\mathcal{F}' \cup \mathcal{F}'')(\Lambda) := \mathcal{F}'(\Lambda) \cup \mathcal{F}''(\Lambda). \tag{3.65}$$

A morphism $\varphi : \Lambda \to \Lambda'$ is mapped by $\mathcal{F}' \cap \mathcal{F}''$ resp. $\mathcal{F}' \cup \mathcal{F}''$ to $\mathcal{F}(\varphi)\big|_{\mathcal{F}'(\Lambda) \cap \mathcal{F}''(\Lambda)}$ resp. $\mathcal{F}(\varphi)\big|_{\mathcal{F}'(\Lambda) \cup \mathcal{F}''(\Lambda)}$.

Clearly, both $\mathcal{F}' \cap \mathcal{F}''$ and $\mathcal{F}' \cup \mathcal{F}''$ are again open subfunctors of $\mathcal{F}$. The functor $\mathrm{emp} : \Lambda \to \emptyset$ is the initial object in $\mathsf{Top}^{\mathsf{Gr}}$.

Definition 3.5.3. *A functor morphism $g : \mathcal{F}'' \to \mathcal{F}$ is called open if there exists a factorization*

$$g : \mathcal{F}'' \xrightarrow{\ f\ } \mathcal{F}' \subset \mathcal{F}$$

such that f is an isomorphism and $\mathcal{F}'$ is an open subfunctor of $\mathcal{F}$.

Finally, the notion of an open covering can be carried over straightforwardly.

Definition 3.5.4. *A family $\{u_\alpha : \mathcal{U}_\alpha \to \mathcal{F}\}$ of open functor morphisms is called an open covering of $\mathcal{F}$ if for each $\Lambda \in \mathsf{Gr}$, the family of maps*

$$u_{\alpha,\Lambda} : \mathcal{U}_\alpha(\Lambda) \to \mathcal{F}(\Lambda)$$

is an open covering of the topological space $\mathcal{F}(\Lambda)$.

It is not hard to check that this assignment of a set of coverings to each functor $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$ endows $\mathsf{Top}^{\mathsf{Gr}}$ with a Grothendieck topology [Mol84], [Sch70], [FGI+05], [AGV64], [MM92], and thus turns it into a *site*.[3]

[3] Categories allowing the definition of a Grothendieck topology are called topoi (or toposes). The choice of a Grothendieck topology turns the topos into a site.

An example of an open subfunctor can be constructed in the following way: let $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$ be an arbitrary functor, and let $U \subset \mathcal{F}(\mathbb{R})$ be an open subset of its underlying set. Then we can construct an open subfunctor $\mathcal{U} \subset \mathcal{F}$ by setting

$$\begin{aligned}\mathcal{U}(\mathbb{R}) &:= U \\ \mathcal{U}(\Lambda) &:= \mathcal{F}(\epsilon_\Lambda)^{-1}(U) \subset \mathcal{F}(\Lambda) \\ \mathcal{U}(\varphi) &:= \mathcal{F}(\varphi)\big|_{\mathcal{U}(\Lambda)} \qquad \text{for } \varphi : \Lambda \to \Lambda'.\end{aligned} \tag{3.66}$$

Here, $\epsilon_\Lambda : \Lambda \to \mathbb{R}$ is the final morphism of $\Lambda \in \mathsf{Gr}$ (cf. (2.35)). It is clear from the definition that the inclusion $\mathcal{U} \subset \mathcal{F}$ is indeed a functor morphism. We will denote subfunctors of this form by $\mathcal{U} = \mathcal{F}\big|_U$ and call them restrictions. It will turn out that in the cases we are interested in, such subfunctors are the only useful ones.

3.5.2 Superdomains and supersmooth morphisms

Now everything is prepared for the introduction of (possibly infinite-dimensional) superdomains.

Definition 3.5.5. *Let $\mathcal{V}$ be a superrepresentable $\overline{\mathbb{K}}$-module in $\mathsf{Top}^{\mathsf{Gr}}$. $\mathcal{V}$ will be called a locally convex, resp. Fréchet, resp. Banach $\overline{\mathbb{K}}$-module if for every $\Lambda \in \mathsf{Gr}$, the topological vector space $\mathcal{V}(\Lambda)$ is locally convex, resp. Fréchet, resp. Banach.*

Definition 3.5.6. *An open subfunctor $\mathcal{F}$ of a locally convex (resp. Fréchet, resp. Banach) $\overline{\mathbb{K}}$-module in $\mathsf{Top}^{\mathsf{Gr}}$ is called a real (or complex, whichever $\overline{\mathbb{K}}$ is) locally convex (resp. Fréchet, resp. Banach) superdomain.*

Definition 3.5.7. *A functor $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$ is called locally isomorphic to real (or complex) locally convex (resp. Fréchet, resp. Banach) superdomains if there exists an open covering $\{u_\alpha : \mathcal{U}_\alpha \to \mathcal{F}\}$ of $\mathcal{F}$ such that each $\mathcal{U}_\alpha$ is a locally convex (resp. Fréchet, resp. Banach) superdomain.*

Restrictions, as it turns out, are the only open subfunctors a superrepresentable $\overline{\mathbb{K}}$-module has.

Proposition 3.5.8. *Any open subfunctor $\mathcal{U} \subset \overline{V}$ of a Banach (resp. Fréchet, resp. locally convex) $\overline{\mathbb{K}}$-module $\overline{V}$ is a restriction $\overline{V}\big|_U$, where $U = \mathcal{U}(\mathbb{K})$ (cf. (3.66)).*

Proof. Clearly, one can write

$$\overline{V} = \overline{V}\big|_{V_{\bar{0}}} \tag{3.67}$$

for a superrepresentable $\overline{\mathbb{K}}$-module $\overline{V}$ which is represented by V, since

$$\overline{V}(\Lambda) = \overline{V}(\epsilon_\Lambda)^{-1}(V_{\bar{0}}). \tag{3.68}$$

Let now $\mathcal{U}$ be an arbitrary open subfunctor of $\overline{V}$ and $U \subset V_{\bar{0}}$ be its $\mathbb{K}$-points. The inclusion $\mathcal{U} \subset \overline{V}$ must be a functor morphism, therefore the diagram

$$\begin{array}{ccc} \mathcal{U}(\Lambda) & \xrightarrow{\subset} & \overline{V}(\Lambda) \\ \big\downarrow{\scriptstyle \mathcal{U}(\epsilon_\Lambda)} & & \big\downarrow{\scriptstyle \overline{V}(\epsilon_\Lambda)} \\ U & \xrightarrow{\subset} & \overline{V}(\mathbb{K}) = V_{\bar{0}} \end{array} \tag{3.69}$$

has to commute for all $\Lambda \in \mathsf{Gr}$. This enforces

$$\mathcal{U}(\Lambda) = \overline{V}(\epsilon_\Lambda)^{-1}(U) \tag{3.70}$$

for all Λ. For any morphism $\varphi : \Lambda \to \Lambda'$ of Grassmann algebras, we also have

$$\begin{array}{ccc} \mathcal{U}(\Lambda) & \xrightarrow{\subset} & \overline{V}(\Lambda) \\ \big\downarrow{\scriptstyle \mathcal{U}(\varphi)} & & \big\downarrow{\scriptstyle \overline{V}(\varphi)} \\ \mathcal{U}(\Lambda') & \xrightarrow{\subset} & \overline{V}(\Lambda') \end{array} \tag{3.71}$$

which commutes again, because the inclusion is a functor morphism. So,

$$\mathcal{U}(\varphi) = \overline{V}(\varphi)\big|_{\mathcal{U}(\Lambda)}. \tag{3.72}$$

□

By the properties of unions and intersections of open subfunctors, we obtain the following corollary.

Corollary 3.5.9. *Let $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$ be a functor which is locally isomorphic to Banach (resp. Fréchet, resp. locally convex) superdomains. Then every open subfunctor $\mathcal{U} \subset \mathcal{F}$ is a restriction $\mathcal{F}\big|_U$.*

Proof. Let $\{u_\alpha : \mathcal{U}_\alpha \to \mathcal{F}\}$ be an open cover of $\mathcal{F}$ by superdomains of the appropriate type and let $\mathcal{U}$ be an arbitrary open subfunctor of $\mathcal{F}$. Then every intersection $\mathcal{U} \cap \mathcal{U}_\alpha$ is a superdomain, i.e. is a restriction $\mathcal{F}\big|_{U \cap U_\alpha}$, where U, U_α are the underlying open sets of the respective functors. Therefore,

$$\mathcal{U}(\Lambda) = \bigcup_\alpha (\mathcal{F}(\epsilon_\Lambda))^{-1}(U_\alpha \cap U) = (\mathcal{F}(\epsilon_\Lambda))^{-1}(U)$$

By the same argument as in Prop. 3.5.8 (functoriality of inclusions), the images of the morphisms of Gr under $\mathcal{U}$ must be the restrictions of those of $\mathcal{F}$. □

It the rest of this Chapter, we will focus exclusivly on Banach superdomains. This is mainly for the reason not to overload the construction with technicalities. After the suitable corrections the constructions described below work equally well in the Freéchet case.

Definition 3.5.10. *Let $\mathcal{V}\big|_U$ and $\mathcal{V}'\big|_{U'}$ be two real Banach superdomains. A functor morphism $f : \mathcal{V}\big|_U \to \mathcal{V}'\big|_{U'}$ is called supersmooth if*

1. *the map*
$$f_\Lambda : \mathcal{V}\big|_U(\Lambda) \to \mathcal{V}'\big|_{U'}(\Lambda) \tag{3.73}$$
is smooth for every $\Lambda \in \mathsf{Gr}$, and

2. *for every $u \in \mathcal{V}\big|_U(\Lambda)$, the derivative*
$$Df_\Lambda(u) : \mathcal{V}(\Lambda) \to \mathcal{V}'(\Lambda) \tag{3.74}$$
is $\Lambda_{\bar{0}}$-linear.

The second condition is necessary and sufficient to turn the sets of differential morphisms

$$\begin{aligned} (Df)_\Lambda : \mathcal{V}\big|_U(\Lambda) \times \mathcal{V}(\Lambda) &\to \mathcal{V}'(\Lambda) &(3.75)\\ (Df)_\Lambda(u,v) &= (Df_\Lambda(u))(v) &(3.76)\end{aligned}$$

into a $\mathcal{V}\big|_U$-family of $\overline{\mathbb{R}}$-linear morphisms $\mathcal{V} \to \mathcal{V}'$.

Together with supersmooth morphisms, smooth Banach superdomains form a category BSDom. Replacing "smooth" with "real analytic" in Def. 3.5.10 leads to the definition of the category of real analytic superdomains.

For complex analytic domains, there seem to be two different approaches. One can start with superdomains in $\mathsf{Top}^{\mathsf{Gr}}$ which are isomorphic to open subfunctors of superrepresentable $\overline{\mathbb{C}}$-modules, and define morphisms to be complex superanalytic functor morphisms. On the other hand, one could as well use the category $\mathsf{Gr}^{\mathbb{C}}$ from the very beginning on, studying only functors in $\mathsf{Top}^{\mathsf{Gr}^{\mathbb{C}}}$ and using morphisms which are analytic in their complex coordinates. However, the two resulting categories are equivalent: the Λ-points of some superrepresentable $\overline{\mathbb{C}}$ module V are $(V \otimes_{\mathbb{R}} \Lambda)_{\bar{0}}$. But

$$(V \otimes_{\mathbb{R}} \Lambda)_{\bar{0}} \cong (V^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \Lambda)_{\bar{0}} \cong (V^{\mathbb{R}} \otimes_{\mathbb{R}} \Lambda^{\mathbb{C}})_{\bar{0}}, \tag{3.77}$$

where $V^{\mathbb{R}}$ is the realification of the super vector space $V = V^{\mathbb{C}}$. Thus, whether one applies analytic maps to the last of these three sets of points, or applies complex analytic ones to the first does not make a difference.

3.5.3 Remark on superdomains of finite differentiability class

In the usual ringed space formalism, it is always stressed that it is impossible to define supermanifolds of class C^k, because the formula (2.50), which is crucial for the definition of supermanifolds, involves an arbitrarily high number of derivatives. There is, however, a way to circumvent this problem: by restricting the category SMan to those whose odd dimension is no greater than $N < \infty$. That seems a bit unnatural, but yields a perfectly consistent theory as well. In the categorical

approach, one simply uses the categories $\mathsf{Gr}_{(N)}$ of Grassmann algebras with no more than N generators. Then the entire theory of superdomains outlined above works just as well with functors from $\mathsf{Sets}^{\mathsf{Gr}_{(N)}}$ and $\mathsf{Top}^{\mathsf{Gr}_{(N)}}$. In fact, working with such functors is even simpler than with the general ones, since one only has to handle a finite number of point sets. In view of formula (2.50), if only N odd dimensions can occur, then the functions F which can still sensibly be pulled back must be of class $C^{\lceil (N+1)/2 \rceil}$. So, in this case, supermanifolds of this class or higher can be consistently defined.

It then becomes meaningful to define superfunctions of finite differentiability class (but at least of class $\lceil (N+1)/2 \rceil$) in the category $\mathsf{SMan}_{(N)}$. It even seems possible to construct "super-Sobolev spaces" of such functions by imposing pointwise a norm and completing C^k-spaces of functions with respect to it [Mol]. With the help of V. Molotkov, I have tried to construct such spaces, and could find no inconsistency in doing this so far. A problem that arises in practical applications of this construction is that one would usually not want to restrict the number of possible odd dimensions. In particular in physical contexts, such a restriction seems unjustifiable. For the solution of the moduli problem later in this work, it was therefore not employed, although it seemed at first that it would enable us to directly carry over the global analytic approach of Tromba [Tro92]. It also seemed not to be any simpler to work with the categories $\mathsf{SMan}_{(N)}$ at first and subsequently let $N \to \infty$. This approach might, however, be viable for certain problems of supergeometry, e.g., for the study of solutions of variational problems involving functionals on supermanifolds, like action functionals of supersymmetric field theories.

3.6 Banach supermanifolds

Actually, the superrepresentable $\overline{\mathbb{K}}$-modules in $\mathsf{Top}^{\mathsf{Gr}}$ do no take their values just in Top, but rather in Man, the category of smooth (resp. complex) Banach manifolds. Since we will only be interested in functors which are locally isomorphic to superdomains, this implies that we may restrict ourselves to functors in the category $\mathsf{Man}^{\mathsf{Gr}}$.

Definition 3.6.1 (Molotkov). *Let $\mathcal{F}$ be a functor in $\mathsf{Man}^{\mathsf{Gr}}$. An open covering $\mathcal{A} = \{u_\alpha : \mathcal{U}_\alpha \to \mathcal{F}\}_{\alpha \in I}$ of $\mathcal{F}$ is called a supersmooth atlas on $\mathcal{F}$ if*

1. *every $\mathcal{U}_\alpha$ is a Banach superdomain,*
2. *for every pair $\alpha, \beta \in I$, the fiber product*
$$\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \times_{\mathcal{F}} \mathcal{U}_\beta \in \mathsf{Man}^{\mathsf{Gr}} \tag{3.78}$$
can be given the structure of a superdomain such that the projections $\Pi_\alpha : \mathcal{U}_{\alpha\beta} \to \mathcal{U}_\alpha$ and $\Pi_\beta : \mathcal{U}_{\alpha\beta} \to \mathcal{U}_\beta$ are supersmooth.

The maps $u_\alpha : \mathcal{U}_\alpha \to \mathcal{F}$ are called charts on $\mathcal{F}$.

Definition 3.6.2. *Two supersmooth atlases $\mathcal{A}, \mathcal{A}'$ on the functor $\mathcal{F} \in \mathsf{Man}^{\mathsf{Gr}}$ are said to be equivalent if their union $\mathcal{A} \cup \mathcal{A}'$ is again a supersmooth atlas on $\mathcal{F}$. A supermanifold $\mathcal{M}$ is a functor in $\mathsf{Man}^{\mathsf{Gr}}$ endowed with an equivalence class of atlases.*

The second condition in Definition 3.6.1 needs some explanation. One might think at first that the fiber product of any two supersmooth superdomains is automatically again supersmooth. But we may only assume here that $\mathcal{F}$ is a functor in $\mathsf{Man}^{\mathsf{Gr}}$, and thus we can a priori only assume the fiber product to exist in $\mathsf{Man}^{\mathsf{Gr}}$. The projection morphisms Π_α, Π_β are therefore only guaranteed to be functor morphisms in $\mathsf{Man}^{\mathsf{Gr}}$. The second condition therefore requires the fiber product of $\mathcal{U}_\alpha, \mathcal{U}_\beta$ to exist in the subcategory $\mathsf{BSDom} \subset \mathsf{Man}^{\mathsf{Gr}}$. Since there do really exist functor isomorphisms in $\mathsf{Man}^{\mathsf{Gr}}$ which are not supersmooth [Mol], this is not automatic.

Definition 3.6.3. *Let $\mathcal{M}, \mathcal{M}'$ be Banach supermanifolds. A functor morphism $f : \mathcal{M} \to \mathcal{M}'$ is called supersmooth if for each pair of charts $u : \mathcal{U} \to \mathcal{M}$, $u' : \mathcal{U}' \to \mathcal{M}'$, the pullback*

$$\begin{tikzcd} \mathcal{U} \times_{\mathcal{M}'} \mathcal{U}' \arrow[rr, "\Pi'"] \arrow[d, "\Pi"'] & & \mathcal{U}' \arrow[d, "u'"] \\ \mathcal{U} \arrow[r, "u"] & \mathcal{M} \arrow[r, "f"] & \mathcal{M}' \end{tikzcd} \tag{3.79}$$

can be given the structure of a Banach superdomain such that its projections Π, Π' are supersmooth.

It is clear that the composition of two supersmooth morphisms of Banach supermanifolds is again supersmooth. Thus Banach supermanifolds form a category BSMan. If nothing else is specified, we will from now on only write SMan for the category of Banach supermanifolds, since it will mainly be them we will be concerned with. The set of supersmooth morphisms $f : \mathcal{M} \to \mathcal{M}'$ will be denoted as $SC^\infty(\mathcal{M}, \mathcal{M}') := \mathrm{Hom}_{\mathsf{BSMan}}(\mathcal{M}, \mathcal{M}')$.

According to Prop. 3.5.8, every open submanifold of a Banach supermanifold $\mathcal{M}$ is of the form $\mathcal{U} = \mathcal{M}\big|_U$, where U is an open submanifold of the underlying manifold $M = \mathcal{M}(\mathbb{K})$. Clearly, the restriction of the supermanifold structure of $\mathcal{M}$ to $\mathcal{U}$ induces on the latter the unique supermanifold structure which makes the inclusion $\mathcal{U} \subset \mathcal{M}$ a supersmooth morphism.

3.6.1 Linear algebra in the category SMan

Evidently, the functor $\bar{\cdot}$ defined in Def. 3.44 takes its values in SMan and not just in $\mathsf{Sets}^{\mathsf{Gr}}$. In addition, if $f : V_1 \times \ldots \times V_n \to V$ is a $\mathbb{K}$-n-linear map of super vector spaces, then $\overline{f} : \overline{V}_1 \times \ldots \times \overline{V}_n \to \overline{V}$ is a supersmooth morphism. Therefore, Corollary 3.4.2 tells us that $\bar{\cdot}$ is a fully faithful functor $\mathsf{SMod}_{\mathbb{K}}(\mathsf{Man}) \to \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{SMan})$. But since all supermanifolds are locally isomorphic to *superrepresentable* $\overline{\mathbb{K}}$-modules, we get more.

Theorem 3.6.4 (see also [Mol84]). *The functor*

$$\bar{\cdot} : \mathsf{SMod}_{\mathbb{K}}(\mathsf{Man}) \to \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{SMan}) \tag{3.80}$$

is an equivalence of categories.

Proof. As seen above, every $\mathbb{K}$-super vector space defines a $\overline{\mathbb{K}}$-module in SMan. Conversely, let a $\overline{\mathbb{K}}$-module $\mathcal{V}$ in SMan be given. It only has to be shown that $\mathcal{V}$ is superrepresentable when considered as a $\overline{\mathbb{K}}$-module in in $\mathsf{Sets}^{\mathsf{Gr}}$. But this follows at once from the fact that each of the sets $\mathcal{V}(\Lambda)$ is already locally isomorphic to the Λ-points $\overline{V}(\Lambda)$ of a superrepresentable module $\overline{V}$. The $\overline{\mathbb{K}}$-module structure on $\mathcal{V}$ then requires all transition functions of the supermanifold to be linear. Therefore, taking any subfunctor of $\mathcal{V}$ which is isomorphic to a superdomain $\overline{V}\big|_U$, we can conclude that $\mathcal{V} \cong \overline{V}$. □

This theorem can be viewed as the categorical version of the statement that supermanifolds must be locally modelled on linear superspaces. This was already obvious in the ringed space formulation for the finite-dimensional case, but Thm. 3.6.4 extends it to the infinite-dimensional case. The analogue of Corollary 3.4.3 also holds.

Corollary 3.6.5. *The functor* $\bar{\cdot} : \mathsf{SMod}_{\mathbb{K}}(\mathsf{Man}) \to \mathsf{SMod}_{\overline{\mathbb{K}}}(\mathsf{SMan})$ *induces equivalences*

$$\bar{\cdot} : \mathsf{SLie}_{\mathbb{K}}(\mathsf{Man}) \to \mathsf{Lie}_{\overline{\mathbb{K}}}(\mathsf{SMan}) \tag{3.81}$$

$$\bar{\cdot} : \mathsf{SAlg}_{\mathbb{K}}(\mathsf{Man}) \to \mathsf{Alg}_{\overline{\mathbb{K}}}(\mathsf{SMan}) \tag{3.82}$$

between the categories of $\mathbb{K}$*-super Lie algebras and ordinary* $\overline{\mathbb{K}}$*-Lie algebras in* SMan*, and between* $\mathbb{K}$*-super algebras and ordinary* $\overline{\mathbb{K}}$*-algebras.*

Proof. This follows immediately from Cor. 3.4.3 and Thm. 3.6.4. □

In fact, it is Thm. 3.6.4 that assures that coherent tensor products and coherent inner Hom-functors exist in the category SMan, since they exist in $\mathsf{SMod}_{\mathbb{K}}(\mathsf{Man})$. For a deeper discussion, see [Mol84] or [Joh02]. Finally, the change of parity functor can be extended to linear supermanifolds.

Definition 3.6.6. *The change of parity functor* $\overline{\Pi}$ *is defined on isomorphism classes of* $\overline{\mathbb{K}}$*-modules in* SMan *as*

$$\overline{\Pi} : \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{SMan}) \to \mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{SMan}) \tag{3.83}$$

$$\mathcal{V} \cong \overline{V} \mapsto \overline{(\Pi(V))}. \tag{3.84}$$

3.6.2 Superpoints, again

Superpoints were introduced in section 2.2.8 as linear supermanifolds corresponding to purely odd super vector spaces. In Prop. 2.2.16, it was shown that the category SPoint is dual to the category Gr, and a duality was chosen, namely

$$\begin{aligned} \mathcal{P} : \mathsf{Gr}^\circ &\to \mathsf{SPoint} \\ \Lambda &\mapsto \operatorname{Spec}(\Lambda) = (\{*\}, \Lambda). \end{aligned} \tag{3.85}$$

In terms of the categorical definition, we have chosen $\mathcal{P}(\Lambda_n) = \overline{\mathbb{K}^{0|n}}$ for the Grassmann algebra on n generators over $\mathbb{K}$. A supermanifold is, in categorical terms, still a functor $\mathsf{Gr} \to \mathsf{Man}$. Thus, $\mathcal{P}$ can be considered as a bifunctor

$$\mathcal{P} : \mathsf{Gr}^\circ \times \mathsf{Gr} \to \mathsf{Man}. \tag{3.86}$$

Proposition 3.6.7. *There exists an isomorphism of bifunctors*

$$\mathcal{P} \cong \operatorname{Hom}_{\mathsf{Gr}}(-,-). \tag{3.87}$$

Proof. We have chosen $\mathcal{P}(\Lambda_n) = \overline{\mathbb{K}^{0|n}}$. Therefore, the Λ_m-points of $\mathcal{P}(\Lambda_n)$ are

$$\mathcal{P}(\Lambda_n)(\Lambda_m) = (\Lambda_m \otimes \mathbb{K}^{0|n})_{\bar{0}} \cong \Lambda_{m,\bar{1}} \otimes \mathbb{K}^n \cong \operatorname{Hom}_{\mathsf{Gr}}(\Lambda_n, \Lambda_m), \tag{3.88}$$

where the last isomorphism was proved in Prop. 2.1.13. □

The following easy fact is also very useful.

Lemma 3.6.8. *There exists an isomorphism of supermanifolds*

$$\mathcal{P}(\mathbb{K}) \times \mathcal{M} \cong \mathcal{M}. \tag{3.89}$$

Proof. Each of the sets of points $\mathcal{P}(\mathbb{K})(\Lambda)$ is just the one-point set:

$$\mathcal{P}(\mathbb{K})(\Lambda) = (\Lambda \otimes \{0\})_{\bar{0}} = \{0\}.$$

Therefore,

$$(\mathcal{P}(\mathbb{K}) \times \mathcal{M})(\Lambda) = \mathcal{P}(\mathbb{K})(\Lambda) \times \mathcal{M}(\Lambda) \cong \mathcal{M}(\Lambda).$$

□

3.7 Connection to the Berezin-Leites theory in the finite-dimensional case

It is instructive to recover the standard ringed space version of a supermanifold from the categorical construction. A rough sketch of the idea can be found in [Mol84]. We will not try to prove every statement here, since we will not need to rely on this construction later on. This section should rather be understood as a heuristic discussion.

One defines an $\overline{\mathbb{R}}$-superalgebra $\mathfrak{R}$ in SMan by setting

$$\begin{aligned} \mathfrak{R}(\Lambda) &:= \Lambda && (3.90)\\ \mathfrak{R}(\varphi) &:= \varphi \quad \text{for } \varphi : \Lambda \to \Lambda'. && (3.91) \end{aligned}$$

The $\overline{\mathbb{R}}$-superalgebra structure is provided by the $\Lambda_{\bar{0}}$-superalgebra structures on each Λ. Note that up to now, we never defined super objects in any of our categories of superobjects. It was never necessary – in fact one of great advantages of the categorical approach is that one can work with purely even objects. This is expressed by Thm. 3.6.4. Every super vector space, superalgebra etc. is an ordinary algebra in $\mathsf{Sets}^{\mathsf{Gr}}$. In this sense $\mathfrak{R}$ is "super super". The functor $\mathfrak{R}$ is still representable as we will show now, but the superalgebra representing it is non-supercommutative.

As an $\overline{\mathbb{R}}$-module, we have

$$\mathfrak{R} \cong \overline{\mathbb{R}} \oplus \overline{\Pi}(\overline{\mathbb{R}}) \cong \overline{\mathbb{R}^{1|1}}. \tag{3.92}$$

We want to find a superalgebra structure on $\mathbb{R}^{1|1}$ which represents the one on $\mathfrak{R}$. Denoting the standard basis of $\mathbb{R}^{1|1}$ as $\{1, \theta\}$, we can write

$$\mathfrak{R}(\Lambda) = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}} \cong \Lambda_{\bar{0}} \otimes 1 \oplus \Lambda_{\bar{1}} \otimes \theta. \tag{3.93}$$

Let $\overline{\mu} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ denote the multiplication in $\mathfrak{R}$ and let μ denote the hypothetical multiplication in $\mathbb{R}^{1|1}$ that we want to determine. Let $\lambda_1, \lambda_2 \in \Lambda_{\bar{0}}$ be given. We have

$$\overline{\mu}_\Lambda(\lambda_1 \otimes 1, \lambda_2 \otimes 1) = \lambda_2 \lambda_1 \otimes \mu(1, 1). \tag{3.94}$$

This must coincide with the product $\lambda_1 \lambda_2 \otimes 1$, by construction of $\mathfrak{R}$. Since λ_1, λ_2 are even, this requires $\mu(1, 1) = 1$. Likewise, for $\lambda_1 \in \Lambda_{\bar{0}}$ and $\lambda_2 \in \Lambda_{\bar{1}}$ we have

$$\overline{\mu}_\Lambda(\lambda_1 \otimes 1, \lambda_2 \otimes \theta) = \lambda_2 \lambda_1 \otimes \mu(1, \theta). \tag{3.95}$$

This must coincide with $\lambda_1 \lambda_2 \otimes \theta$, and thus we must have $\mu(1, \theta) = \theta$. Analogously we find $\mu(\theta, 1) = \theta$.

Let then $\lambda_1, \lambda_2 \in \Lambda_{\bar{1}}$ be given. The product is

$$\overline{\mu}_\Lambda(\lambda_1 \otimes \theta, \lambda_2 \otimes \theta) = \lambda_2 \lambda_1 \otimes \mu(\theta, \theta). \tag{3.96}$$

This must be equal to $\lambda_1 \lambda_2 \otimes 1$, which enforces $\mu(\theta, \theta) = -1$.

The super space $\mathbb{R}^{1|1}$ endowed with this multiplication will be denoted as $\mathbb{C}^s$. As an $\mathbb{R}$-algebra it is isomorphic to $\mathbb{C}$, but as an $\mathbb{R}$-superalgebra, it is isomorphic to $\mathbb{C}$ with i declared odd. It is non-supercommutative, in particular, every non-zero element is invertible (which makes it a kind of super analog of a skew field). Note that, although $\mathbb{C}^s$ is non-supercommutative, the $\overline{\mathbb{R}}$-superalgebra $\mathfrak{R}$ *is* supercommutative. Just like passing to their functors of points turns supercommutative algebras into commutative $\overline{\mathbb{R}}$-algebras, the special non-supercommutativity of $\mathbb{C}^s$ is weakened to supercommutativity of its functor of points as an $\overline{\mathbb{R}}$-algebra. It

would be an interesting question to study whether this kind of reasoning can be iterated to yield something like "super super super" objects and whether these would possess any geometric interpretation. Molotkov in [Mol84] proposes a formalism to investigate such questions, but a conclusive answer has yet to be found.

The reason why we introduced $\mathfrak{R}$ is that we need a superalgebra in SMan in order to induce the structure of a superalgebra on certain sets of morphisms which we want to interpret as the superfunctions on a supermanifold $\mathcal{M}$. Following Molotkov [Mol84], we define an $\mathbb{R}$-superalgebra

$$SC^\infty(\mathcal{M}) := SC^\infty(\mathcal{M}, \mathfrak{R}), \tag{3.97}$$

which will be called the superalgebra of superfunctions on $\mathcal{M}$. Since $\mathfrak{R}$ is a supercommutative $\overline{\mathbb{R}}$-superalgebra, $SC^\infty(\mathcal{M})$ is canonically equipped with the structure of a supercommutative $SC^\infty(\mathcal{M}, \overline{\mathbb{R}})$-superalgebra. Moreover, we can embed $\mathbb{R} \hookrightarrow SC^\infty(\mathcal{M})$ as the constant functions $\mathcal{M} \to \mathbb{R}$. More precisely, for any $r \in \mathbb{R}$, we define a morphism $f_r : \mathcal{M} \to \mathbb{R}$ by setting

$$(f_r)_\Lambda(m) = r \qquad \text{for all } m \in \mathcal{M}(\Lambda). \tag{3.98}$$

These are obviously supersmooth morphisms. Via this embedding, $SC^\infty(\mathcal{M}, \mathfrak{R})$ becomes endowed with an $\mathbb{R}$-superalgebra structure.

The following example is borrowed from [Mol84]. Consider a superdomain $\mathcal{U} \subset \overline{\mathbb{R}^{m|n}}$. One has maps

$$\begin{aligned} x_i : \overline{\mathbb{R}^{m|n}} &\to \overline{\mathbb{R}^{1|0}} \hookrightarrow \mathfrak{R}, \qquad 1 \le i \le m, & (3.99)\\ \theta_j : \overline{\mathbb{R}^{m|n}} &\to \overline{\mathbb{R}^{0|1}} \hookrightarrow \mathfrak{R}, \qquad 1 \le j \le n, & (3.100)\end{aligned}$$

where the first arrow in every line represents the canonical projection onto the i-th even and j-th odd coordinate, respectively. As in ordinary geometry, the sheaf of functions can be generated from these coordinate maps. One can show that [Mol84]

$$SC^\infty(\mathcal{U}) \cong C^\infty(x_1, \ldots, x_m) \otimes \wedge^\bullet[\theta_1, \ldots, \theta_n], \tag{3.101}$$

where $C^\infty(x_1, \ldots, x_m) = C^\infty_U$.

Now let $\mathcal{M}$ be a supermanifold and let M be its underlying topological space (i.e., the topological space underlying the base manifold $\mathcal{M}(\mathbb{R})$). Then we can assign to every open set $U \subset M$ the $\mathbb{R}$-superalgebra $SC^\infty(\mathcal{M}\big|_U)$. This yields a presheaf $P(\mathcal{M})$ on M. The standard procedure of "sheafification", i.e., taking for every point $x \in M$ the direct limit over all open sets containing it produces the associated sheaf $S(\mathcal{M})$. It is clear that any morphism $f : \mathcal{M} \to \mathcal{M}'$ of supermanifolds induces, via its associated map $M \to M'$ of the underlying spaces, a morphism of sheaves $S(f) : S(\mathcal{M}) \to S(\mathcal{M}')$. Therefore the assignment

$$\begin{aligned} \mathcal{S} : \mathcal{M} &\mapsto S(\mathcal{M}) & (3.102)\\ f &\mapsto S(f) & (3.103)\end{aligned}$$

defines a functor from the category of supermanifolds to the category of topological spaces locally ringed by supercommutative superalgebras.

Denote by FinSMan the category of finite-dimensional supermanifolds defined by the categorical construction. Moreover, we call the category of finite-dimensional topological spaces locally ringed by supercommutative superalgebras the category of Berezin-Leites supermanifolds. Then we have the following:

Theorem 3.7.1 (Molotkov [Mol84])**.** *The functor $\mathcal{S}$ establishes an equivalence between the category* FinSMan *and the category of Berezin-Leites supermanifolds.*

We do not want to go into the proof of this Theorem here. Let us just point out that it really only holds in the finite-dimensional case, for the same reasons as in ordinary geometry: while in any finite dimension $m|n$, there exists up to isomorphy only one super vector space to which a supermanifold can be identified (in the functor of points sense), this is no longer the case in infinite dimensions.

3.8 Super vector bundles

Following Molotkov [Mol84], we will define super vector bundles, like supermanifolds, in terms of atlases of trivial super vector bundles.

Definition 3.8.1. *A smooth trivial super vector bundle over $\mathcal{M}$ is defined to be a triple $(\mathcal{M}\times\mathcal{V},\mathcal{M},\Pi_{\mathcal{M}})$, where $\mathcal{M}$ is a supermanifold, $\mathcal{V}$ is a superrepresentable $\overline{\mathbb{R}}$-module and $\Pi_{\mathcal{M}} : \mathcal{M}\times\mathcal{V}\to\mathcal{M}$ is the canonical projection. A morphism $(\mathcal{M}\times\mathcal{V},\mathcal{M},\Pi_{\mathcal{M}})\to(\mathcal{M}'\times\mathcal{V}',\mathcal{M}',\Pi_{\mathcal{M}'})$ of trivial super vector bundles consists of a pair of supersmooth morphisms*

$$\begin{aligned} f:\mathcal{M}\times\mathcal{V} &\to \mathcal{M}'\times\mathcal{V}' \quad (3.104)\\ g:\mathcal{M} &\to \mathcal{M}', \quad (3.105)\end{aligned}$$

such that $\Pi_{\mathcal{M}'}\circ f = g\circ\Pi_{\mathcal{M}}$, and such that $\Pi_{\mathcal{V}'}\circ f:\mathcal{M}\times\mathcal{V}\to\mathcal{V}'$ is a $\mathcal{M}$-family of $\overline{\mathbb{R}}$-linear morphisms (cf. definition 3.2.5).

By definition, trivial super vector bundles are therefore particular functors $\mathsf{Gr}\to\mathsf{VBun}$, where VBun is the category of smooth vector bundles. Every functor $\mathcal{E}\in\mathsf{VBun}^{\mathsf{Gr}}$ gives rise to a functor $\mathcal{M}:\mathsf{Gr}\to\mathsf{Man}$ in a canonical way: every bundle $\mathcal{E}(\Lambda)$ possesses a projection map $\pi(\Lambda):\mathcal{E}(\Lambda)\to M(\Lambda)$, where $M(\Lambda)$ is an ordinary manifold. Then, clearly, setting $\mathcal{M}(\Lambda)=M(\Lambda)$ defines a functor in $\mathsf{Man}^{\mathsf{Gr}}$.

Definition 3.8.2. *Let $\mathcal{E},\mathcal{E}'$ be functors in $\mathsf{VBun}^{\mathsf{Gr}}$, and let $\mathcal{M},\mathcal{M}'$ be their associated functors in $\mathsf{Man}^{\mathsf{Gr}}$. Then $\mathcal{E}$ is said to be an open subfunctor of $\mathcal{E}'$, denoted $\mathcal{E}\subset\mathcal{E}'$, if*

1. *$\mathcal{M}$ is an open subfunctor of $\mathcal{M}'$, and*
2. *for each $\Lambda\in\mathsf{Gr}$ we have $\pi(\Lambda)^{-1}(\mathcal{M}(\Lambda))=\pi'(\Lambda)^{-1}(\mathcal{M}(\Lambda))$,*

where $\pi(\Lambda)$ is the projection of the bundle $\mathcal{E}(\Lambda)$ to its base $\mathcal{M}(\Lambda)$.

This gives us again the notion of an open morphism $\mathcal{E}'' \to \mathcal{E}$ of functors in $\mathsf{VBun}^{\mathsf{Gr}}$: it is called open if it can be factorized as a composition

$$\mathcal{E}'' \xrightarrow{\ f\ } \mathcal{E}' \subset \mathcal{E}, \tag{3.106}$$

where f is an isomorphism of functors and $\mathcal{E}'$ is an open subfunctor of $\mathcal{E}$. An open covering $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ of a $\mathcal{E} \in \mathsf{VBun}^{\mathsf{Gr}}$ is then a collection of open morphisms $\{\phi_\alpha : \mathcal{E}_\alpha \to \mathcal{E}\}_{\alpha\in A}$, such that the associated maps $\{\pi\circ\phi_\alpha\}_{\alpha\in A}$ are an open covering of the functor $\mathcal{M} : \mathsf{Gr} \to \mathsf{Man}$ associated with $\mathcal{E}$. In analogy to supermanifolds, a supervector bundle is a functor in $\mathsf{VBun}^{\mathsf{Gr}}$ endowed with an atlas of trivial open subbundles.

Definition 3.8.3. *Let $\mathcal{E}$ be a functor in $\mathsf{VBun}^{\mathsf{Gr}}$, and let $\mathcal{M} \in \mathsf{Man}^{\mathsf{Gr}}$ be its associated functor of base manifolds. Let $\mathcal{A} = \{\phi_\alpha : \mathcal{E}_\alpha \to \mathcal{E}\}_{\alpha\in A}$ be an open covering of $\mathcal{E}$. Then this covering is an atlas of a super vector bundle $\mathcal{E}$ over the supermanifold $\mathcal{M}$ if the following conditions hold:*

1. *each of the $\mathcal{E}_\alpha$ is a trivial super vector bundle $\mathcal{U}_\alpha \times \mathcal{V}_\alpha$, and $\mathcal{V}_\alpha \cong \mathcal{V}_\beta$ for all $\alpha, \beta \in A$, and*
2. *for each $\alpha, \beta \in A$, the overlaps*

$$\begin{array}{ccc} \mathcal{E}_\alpha \times_{\mathcal{E}} \mathcal{E}_\beta & \xrightarrow{\Pi_\alpha} & \mathcal{E}_\alpha \\ {\scriptstyle \Pi_\beta}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_\alpha} \\ \mathcal{E}_\beta & \xrightarrow{\phi_\beta} & \mathcal{E} \end{array} \tag{3.107}$$

can be given the structure of a trivial super vector bundle in such a way that the projections Π_α, Π_β become morphisms of trivial super vector bundles.

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are equivalent, if their union $\mathcal{A} \cup \mathcal{A}'$ is again an atlas. A super vector bundle $\mathcal{E}$ is a functor in $\mathsf{VBun}^{\mathsf{Gr}}$ together with an equivalence class of atlases.

The second condition is necessary because the fiber product in the diagram is constructed as the fiber product in $\mathsf{VBun}^{\mathsf{Gr}}$. We thus have to make sure that it actually exists in the subcategory of trivial super vector bundles (compare to the discussion following Def. 3.6.1). Note also that the requirement that the transition functions be morphisms of trivial super vector bundles automatically turns $\mathcal{M}$ into a supermanifold. In general, one would usually start with a given base supermanifold and construct a super vector bundle on it by choosing a local trivialization which is compatible with the transition functions of the base.

Definition 3.8.4. *Let $\mathcal{E}, \mathcal{E}'$ be super vector bundles with open coverings $\{\phi_\alpha : \mathcal{E}_\alpha \to \mathcal{E}\}_{\alpha\in A}$ and $\{\phi_{\alpha'} : \mathcal{E}'_{\alpha'} \to \mathcal{E}'\}_{\alpha'\in A'}$. A functor morphism $\Phi : \mathcal{E} \to \mathcal{E}'$ in*

$\mathsf{VBun}^{\mathsf{Gr}}$ *is a morphism of super vector bundles if for all* $\alpha \in A$ *and all* $\alpha' \in A'$*, the pullbacks*

$$\begin{tikzcd} \mathcal{E}_\alpha \times_{\mathcal{E}'} \mathcal{E}_{\alpha'} \arrow[rr, "\Pi_{\alpha'}"] \arrow[d, "\Pi_\alpha"'] & & \mathcal{U}_{\alpha'} \arrow[d, "\phi_{\alpha'}"] \\ \mathcal{U}_\alpha \arrow[r, "\phi_\alpha"] & \mathcal{E} \arrow[r, "\Phi"] & \mathcal{E}' \end{tikzcd} \tag{3.108}$$

can be chosen such that $\mathcal{E}_\alpha \times_{\mathcal{E}'} \mathcal{E}_{\alpha'}$ *is a trivial super vector bundle and the projections* $\Pi_\alpha, \Pi_{\alpha'}$ *are morphisms of trivial super vector bundles.*

Definitions 3.8.3 and 3.8.4 yield a category SVBun, which is obviously a subcategory of $\mathsf{VBun}^{\mathsf{Gr}}$, but not a full one. One can define super vector bundles in terms of cocycles with values in a Lie supergroup as well [Mol84], but we will not attempt to do this here.

We have a natural functor

$$\begin{aligned} B : \mathsf{SVBun} &\to \mathsf{SMan} && (3.109)\\ (\pi : \mathcal{E} \to \mathcal{M}) &\mapsto \mathcal{M} && (3.110) \end{aligned}$$

which assigns to every super vector bundle its base supermanifold. The resulting full subcategories of bundles are denoted by $\mathsf{SVBun}(\mathcal{M}) := B^{-1}(\mathcal{M})$.

Proposition 3.8.5. *A super vector bundle* $\pi : \mathcal{E} \to \mathcal{M}$ *is trivial if and only if all of its* Λ*-points* $\pi_\Lambda : \mathcal{E}(\Lambda) \to \mathcal{M}(\Lambda)$ *are trivial bundles.*

Proof. The bundle $\pi : \mathcal{E} \to \mathcal{M}$ is trivial if and only if there exists an isomorphism $f : \mathcal{E} \to \mathcal{M} \times \mathcal{V}$ for some superrepresentable $\mathbb{K}$-module $\mathcal{V}$ such that $\pi = \pi_{\mathcal{M}} \circ f$. This means that for every $\Lambda \in \mathsf{Gr}$, the components of f must make the diagram

$$\begin{tikzcd} f_\Lambda : \mathcal{E}(\Lambda) \arrow[rr] \arrow[dr, "\pi_\Lambda"'] & & \mathcal{M}(\Lambda) \times \mathcal{V}(\Lambda) \arrow[dl, "\pi_{\mathcal{M},\Lambda}"] \\ & \mathcal{M}(\Lambda) & \end{tikzcd} \tag{3.111}$$

commutative. That is precisely the condition for the triviality of the ordinary vector bundle $\pi_\Lambda : \mathcal{E}(\Lambda) \to \mathcal{M}(\Lambda)$. □

3.8.1 Pullback of super vector bundles

Let $\pi : \mathcal{E} \to \mathcal{M}$ be a super vector bundle, and let $f : \mathcal{M}' \to \mathcal{M}$ be a supersmooth morphism of supermanifolds. Then we define the pullback of $\mathcal{E}$ along f as the functor

$$\begin{aligned} f^*\mathcal{E} : \mathsf{Gr} &\to \mathsf{VBun} && (3.112)\\ \Lambda &\mapsto f_\Lambda^*(\mathcal{E}(\Lambda)). && (3.113) \end{aligned}$$

It is clear that by this construction, $f^*\mathcal{E}$ is again a super vector bundle: pulling back each of the trivial open subbundles $\mathcal{E}_\alpha$ which make up the atlas of $\mathcal{E}$ gives an atlas on $f^*\mathcal{E}$ [Mol84]. It is equally clear that the pullback bundle thus defined is indeed the pullback in the homological sense: it completes the cartesian square

$$\begin{array}{ccc} f^*\mathcal{E} & \xrightarrow{\Pi_\mathcal{E}} & \mathcal{E} \\ \Pi_{f^*\mathcal{E}}\big\downarrow & & \big\downarrow\pi \\ \mathcal{M}' & \xrightarrow{f} & \mathcal{M} \end{array} \quad , \tag{3.114}$$

where $\Pi_\mathcal{E}$ is the canonical projection defined pointwise by the pullback bundles $f^*_\Lambda\mathcal{E}(\Lambda)$. As in the ordinary case, any given morphism $f : \mathcal{M}' \to \mathcal{M}$ gives in this way rise to a functor

$$f^* : \mathsf{SVBun}(\mathcal{M}) \to \mathsf{SVBun}(\mathcal{M}'). \tag{3.115}$$

In spite of the formal similarity, it is somewhat dangerous to think of the $\overline{\mathbb{R}}$-modules $\mathcal{V}$ in $\Pi_\mathcal{M} : \mathcal{M} \times \mathcal{V} \to \mathcal{M}$ as "fibers" in the set-theoretical sense, since they are parametrized not just by the "points", i.e., the underlying manifold of $\mathcal{M}$, but by the odd dimensions of $\mathcal{M}$ as well. However, this becomes true again on the underlying manifold: looking at an underlying point $x : \mathcal{P}(\mathbb{R}) \to \mathcal{M}$ of $\mathcal{M}$, we note that, since each $\mathcal{P}(\mathbb{R})(\Lambda)$ consists of a single element,

$$x^*\mathcal{E} \cong \mathcal{V}, \tag{3.116}$$

where $\mathcal{V}$ is the typical fiber of $\mathcal{E}$, i.e. a superrepresentable $\overline{\mathbb{R}}$-module. This means that the pullback of $\mathcal{E}$ along the inclusion $M_{red} \hookrightarrow \mathcal{M}$ yields a canonical $\mathbb{Z}_2$-graded vector bundle over the ordinary manifold M_{red}.

3.8.2 The tangent bundle $\mathcal{TM}$ of a supermanifold $\mathcal{M}$

The tangent bundle $\mathcal{TM}$ of a supermanifold $\mathcal{M}$ is defined in the categorical framework as a functor $\mathcal{TM} : \mathsf{Gr} \to \mathsf{VBun}$ in the following way: for every $\Lambda \in \mathsf{Gr}$ and every $\varphi : \Lambda \to \Lambda'$, set

$$\begin{aligned} \mathcal{TM}(\Lambda) &:= T(\mathcal{M}(\Lambda)), \\ \mathcal{TM}(\varphi) &:= D(\mathcal{M}(\varphi)) : T(\mathcal{M}(\Lambda)) \to T(\mathcal{M}(\Lambda')). \end{aligned} \tag{3.117}$$

To every morphism $f : \mathcal{M} \to \mathcal{M}'$ of supermanifolds, we assign a functor morphism

$$\begin{aligned} \mathcal{D}f : \mathcal{TM} &\to \mathcal{TM}' \\ (\mathcal{D}f)_\Lambda &:= Df_\Lambda : T(\mathcal{M}(\Lambda)) \to T(\mathcal{M}'(\Lambda)). \end{aligned} \tag{3.118}$$

The assignments (3.117) and (3.118) define a functor $\mathcal{T} : \mathsf{SMan} \to \mathsf{VBun}^{\mathsf{Gr}}$ which will be called the tangent functor. For the definition of a super vector bundle to make sense, we would certainly expect the tangent bundle to be in SVBun, not just in $\mathsf{VBun}^{\mathsf{Gr}}$. This is indeed the case:

Proposition 3.8.6. *The tangent functor is a functor* $\mathcal{T} : \mathsf{SMan} \to \mathsf{SVBun}$.

Proof. Choose a supersmooth atlas $\{u_\alpha : \mathcal{U}_\alpha \to \mathcal{M}\}_{\alpha \in A}$ of $\mathcal{M}$ which satisfies the conditions of Def. 3.6.1. Then all $\mathcal{U}_\alpha$ are open domains in some superrepresentable $\overline{\mathbb{R}}$-module $\mathcal{V}$, so their tangent bundles are trivial and isomorphic to

$$\mathcal{T}\mathcal{U}_\alpha \cong \mathcal{U}_\alpha \times \mathcal{V}. \tag{3.119}$$

It is clear that the tangent bundles $\{\mathcal{T}\mathcal{U}_\alpha\}_{\alpha \in A}$ of the coordinate domains form an atlas of open subfunctors for the functor $\mathcal{T}\mathcal{M} \in \mathsf{VBun}^{\mathsf{Gr}}$. It has to be shown that this atlas satisfies the conditions of Def. 3.8.3.

By Definition 3.6.1, each intersection $\mathcal{U}_\alpha \times_{\mathcal{M}} \mathcal{U}_\beta$ has the structure of a superdomain itself, and the projections to $\mathcal{U}_\alpha, \mathcal{U}_\beta$ are supersmooth. Thus, the intersection $\mathcal{T}\mathcal{U}_\alpha \times_{\mathcal{T}\mathcal{M}} \mathcal{T}\mathcal{U}_\beta$ has the structure of a trivial super vector bundle as well. Moreover, the maps $\mathcal{D}\Pi_\alpha$ and $\mathcal{D}\Pi_\beta$ are $\mathcal{U}_\alpha \times_{\mathcal{M}} \mathcal{U}_\beta$-families of morphisms of $\overline{\mathbb{R}}$-modules (cf. definition 3.5.10). Therefore, we have the commutative square

$$\begin{tikzcd} \mathcal{T}\mathcal{U}_\alpha \times_{\mathcal{T}\mathcal{M}} \mathcal{T}\mathcal{U}_\beta \arrow[r, "\mathcal{D}\Pi_\beta"] \arrow[d, "\mathcal{D}\Pi_\alpha"'] & \mathcal{T}\mathcal{U}_\beta \arrow[d] \\ \mathcal{T}\mathcal{U}_\alpha \arrow[r] & \mathcal{T}\mathcal{M} \end{tikzcd}, \tag{3.120}$$

which is precisely the second condition of 3.8.3. □

One can also show [Mol84] that the sections of the tangent bundle $\mathcal{T}\mathcal{M}$ thus constructed have the properties of vector fields in the sense that an action of these sections on the sections of any "natural bundle" (like functions, tensor fields, etc.) can be defined which possesses the properties of the Lie derivative [GLS02].

3.8.3 The endomorphism bundle $\mathcal{E}nd(\mathcal{E})$ of a super vector bundle

To construct bundles of morphisms, it is most convenient to use the intuition of Section 3.3.1, in particular Prop. 3.3.2. In the spirit of the family point of view, one should consider a super vector bundle $\pi : \mathcal{E} \to \mathcal{M}$ as an object in the category $\mathsf{SMan}/\mathcal{M}$. The fact that it is a vector bundle and not an arbitrary fiber bundle is then translated into an $\mathcal{M}^*(\overline{\mathbb{R}}) = \mathcal{M} \times \overline{\mathbb{R}}$ action on $\mathcal{E}$ (compare to Section 3.3.1). The dual bundle $\mathcal{E}^*$ is defined to be

$$\mathcal{E}^* := \underline{\mathrm{Hom}}_{\mathsf{SVBun}(\mathcal{M})}(\mathcal{E}, \mathcal{M}^*(\overline{\mathbb{R}})), \tag{3.121}$$

i.e., it is an inner Hom-object in the category of super vector bundles over $\mathcal{M}$. In the same vein, one defines the endomorphism bundle of $\mathcal{E}$.

Definition 3.8.7. *Let* $\pi : \mathcal{E} \to \mathcal{M}$ *be a super vector bundle. The endomorphism bundle of* $\mathcal{E}$ *is defined to be the inner Hom-object*

$$\mathcal{E}nd(\mathcal{E}) := \underline{\mathrm{Hom}}_{\mathsf{SVBun}(\mathcal{M})}(\mathcal{E}, \mathcal{E}). \tag{3.122}$$

The existence of inner Hom-objects in the categories $\mathsf{SVBun}/\mathcal{M}$ can be deduced from their existence in SVec and local triviality, but for nontrivial super vector bundles, this is a rather tedious job. We take these objects for granted, relying on the work of Molotkov [Mol84], [Mol99] and the general theory of topoi, expounded for example in [Joh02] and [MM92].

3.8.4 The change of parity functor $\overline{\Pi}$ for super vector bundles

The functor $\overline{\Pi}$ defined in Definitions 3.4.5 and 3.6.6 extends to super vector bundles. For any trivial super vector bundle $\mathcal{M} \times \mathcal{V}$, we set

$$\overline{\Pi}(\mathcal{M} \times \mathcal{V}) := \mathcal{M} \times \overline{\Pi}(\mathcal{V}). \tag{3.123}$$

To show that an arbitrary super vector bundle $\pi : \mathcal{E} \to \mathcal{M}$ gets mapped into a well defined parity-reversed version $\overline{\Pi}(\mathcal{E})$, note first that for any pair V, V' of $\mathbb{R}$-super vector spaces, one has the natural isomorphism

$$\mathrm{Hom}_{\mathsf{SVec}_{\mathbb{R}}}(V, V') \cong \mathrm{Hom}_{\mathsf{SVec}_{\mathbb{R}}}(\Pi(V), \Pi(V')), \tag{3.124}$$

which entails

$$\underline{\mathrm{Hom}}_{\mathsf{SVec}_{\mathbb{R}}}(V, V') = \mathcal{L}_{\overline{\mathbb{R}}}(V; V') \cong \mathcal{L}_{\overline{\mathbb{R}}}(\Pi(V); \Pi(V')). \tag{3.125}$$

As remarked above, Thm. 3.6.4 ensures the existence of inner Hom-objects in the category $\mathsf{Mod}_{\overline{\mathbb{K}}}(\mathsf{SMan})$. This allows us to extend equation (3.125) to an isomorphism of families of $\overline{\mathbb{R}}$-linear maps (cf. Sections 3.2.2 and 3.2.3)

$$L_{\overline{\mathbb{R}}}(\mathcal{M}; \mathcal{V}; \mathcal{V}') \cong L_{\overline{\mathbb{R}}}(\mathcal{M}; \Pi(\mathcal{V}); \Pi(\mathcal{V}')). \tag{3.126}$$

This is precisely what we need, because $L_{\overline{\mathbb{R}}}(\mathcal{M}; \mathcal{V}; \mathcal{V}')$ is, of course, isomorphic to the set of morphisms of trivial super vector bundles $\mathrm{Hom}_{\mathsf{SVBun}}(\mathcal{M} \times \mathcal{V}, \mathcal{M} \times \mathcal{V}')$.

Let now $\pi : \mathcal{E} \to \mathcal{M}$ be an aribtrary super vector bundle, and let $\{\mathcal{E}_\alpha\}_{\alpha \in A}$ be an atlas of trivial super vector bundles. Then we can just take the atlas $\{\Pi(\mathcal{E}_\alpha)\}_{\alpha \in A}$ of parity reversed bundles to define the bundle $\Pi(\mathcal{E})$. This atlas satisfies the conditions of Definition 3.8.3: to each overlap $\mathcal{E}_\alpha \times_{\mathcal{E}} \mathcal{E}_\beta$, one assigns its parity reversed counterpart, and to each of the projections $\Pi_\alpha : \mathcal{E}_\alpha \times_{\mathcal{E}} \mathcal{E}_\beta \to \mathcal{E}_\alpha$, one assigns its image under the isomorphism (3.126). For a more detailed and formal exposition, cf. [Mol99], [Mol84].

Chapter 4

Superconformal surfaces

The widely used term *super Riemann surface* suggests that there should exist a unique generalization of Riemann surfaces to supergeometry. This is not at all the case. Not all of the features of a Riemann surface have a unique super analog. In particular, there are infinitely many types of superconformal structures (four families and several exceptional ones) which can serve as super analogs of the conformal structure of a Riemann surface. Superconformal structures are not the same as supercomplex structures anymore, instead, a supercomplex structure is a particular superconformal structure.

The super Riemann surfaces used in superstring theory are only one of these infinitely many kinds of superconformal surfaces, whose specific features make it especially useful for physics. They are the surfaces associated to the algebra $\mathfrak{k}^L(1|1)$, which can be seen as a $1|1$-dimensional analog of the contact vector fields. In particular, the algebra $\mathfrak{k}^L(1|1)$ possesses a non-trivial central extension, which is an indispensible ingredient of the field theory constructed on it.

4.1 The superconformal algebras

The history of the superalgebras which are nowadays subsumed under the term "superconformal" reaches back to the days of the so-called dual models, a topic in the study of hadronic and mesonic resonances whose ideas and methods directly led to the emergence of string theory during the seventies. The superconformal algebras comprise four main series, accompanied by several exceptional ones. The main series were discovered in the early seventies, first by Neveu and Schwarz [NS71] and Ramond [Ram71], and then by Ademollo, Brink et al. [ABD$^+$76b], [ABD$^+$76a]. Schwimmer and Seiberg later found an additional one-parameter family of deformations for one type of these series. The exceptional ones were announced in [Shc99], [GLS05]. The completeness of the classification by the four series was conjectured in [KvdL89], and their central extensions were completely classified. An extended list, including the exceptional algebras and several additional extensions, was conjectured to be the complete classification in [GLS05] and

seems to be the definitive answer today. The authors of [GLS05] also argue for the use of the term "stringy" instead of "superconformal", since the latter seems to suggest that these algebras preserve some geometric structure up to a factor. But only a few of them indeed do that, while the others merely inherit their name from containing the algebra of conformal transformations of $\mathbb{C}^\times$.

4.1.1 Definition

Although the geometric structures whose moduli space we want to construct in this work are preserved by only two particular superconformal algebras, we will give a brief overview of their definitions and explicit realizations in general in this section. Each of these algebras defines its own species of superconformal structure on a supersurface, thus each of them could rightfully be associated with its own kind of super Riemann surface. Each of them therefore poses a moduli problem, of which the one studied in this work (and dubbed "the" super moduli space in many physics papers) is just one particular case.

We study the complex punctured superspace $\overline{\mathbb{C}^{1|n}} \setminus \{0\}$, i.e., the superspace $\overline{\mathbb{C}^{1|n}}$ with the stalk at zero removed. Denote by $(z, \theta_1, \ldots, \theta_n)$ its coordinates and by $\mathcal{P}^L(n) := \mathbb{C}[z, z^{-1}, \theta_1, \ldots, \theta_n]$ the algebra of Laurent polynomials on $\overline{\mathbb{C}^{1|n}} \setminus \{0\}$. Then the algebra $\operatorname{der}(\mathcal{P}^L(n))$ of its derivations is

$$\mathfrak{vect}^L(1|n) = \left\{ X = f_0 \frac{\partial}{\partial z} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial \theta_i} \mid f_i \in \mathcal{P}^L(n),\ 0 \leq i \leq n \right\}, \tag{4.1}$$

the algebra of vector fields on the punctured complex superplane with Laurent coefficients. All superconformal algebras can be realized as subalgebras of $\mathfrak{vect}^L(1|n)$.

Definition 4.1.1. *The Witt algebra (or centerless Virasoro algebra)* $\mathfrak{witt}$ *is the algebra*

$$\operatorname{der}(\mathcal{P}^L(0)) = \mathfrak{vect}^L(1|0) = \left\{ X = f(z) \frac{\partial}{\partial z} \mid f(z) \in \mathcal{P}^L(0) \right\} \tag{4.2}$$

of derivations of the Laurent polynomials on $\mathbb{C}^\times$.

The following definition is borrowed from [KvdL89].

Definition 4.1.2. *A Lie superalgebra* $\mathfrak{g}$ *is called superconformal, if*

1. $\mathfrak{g}$ *is simple,*
2. $\mathfrak{g}$ *contains* $\mathfrak{witt}$ *as a subalgebra, and*
3. $\mathfrak{g}$ *has growth 1.*

The third condition means that whenever one takes a finite set of elements $x_1, \ldots, x_k \in \mathfrak{g}$ and computes the linear span V_j of commutators of the x_i of length $\leq j$, then $\dim V_j \leq C(x_1, \ldots, x_k) \cdot j$, where C is a constant independent of j.

Since it will turn out that every superconformal algebra is a subalgebra of some $\mathfrak{vect}^L(1|n)$, each of them has a standard $\mathbb{Z}$-grading, but some have additional nonstandard gradings. Actually, the definition of stringy superalgebras given in [GLS05] slightly extends the one of superconformal algebras above. The authors of [GLS05] carry over a definition of O. Mathieu, who introduced the notion of a *deep algebra* for simple $\mathbb{Z}$-graded Lie algebras [Mat92] to the case of superalgebras. Then an infinitely deeply $\mathbb{Z}$-graded superalgebra is called *stringy*, according to [GLS05], if it possesses a root vector which does not act locally nilpotently. This will be the case for all superconformal algebras (after Def. 4.1.2), since they have the root vector $\frac{\partial}{\partial z}$. A detailed account can be found in [GLS05] and the references therein. For our purposes, it will suffice to just outline the explicit construction of the superconformal algebras and the geometric structures they preserve.

4.1.2 The algebras $\mathfrak{vect}^L(1|n)$ and $\mathfrak{svect}^L(1|n)$

Obviously, the algebras $\mathfrak{vect}^L(1|n)$ are all superconformal. They comprise the first series of those algebras and can be viewed as the algebra of holomorphic vector fields on $\overline{\mathbb{C}^{1|n}} \setminus \{0\}$.

The second series is the divergence free vector fields. For a vector field $X = f_0\frac{\partial}{\partial z} + \sum_{i=1}^n f_i\frac{\partial}{\partial\theta_i} \in \mathfrak{vect}^L(1|n)$, its divergence is defined as

$$\operatorname{div} X = \frac{\partial f_0}{\partial z} + \sum_{i=1}^n (-1)^{p(f_i)}\frac{\partial f_i}{\partial\theta_i}. \tag{4.3}$$

Then the algebras

$$\mathfrak{svect}^L_\lambda(1|n) := \{X \in \mathfrak{vect}^L(1|n) \mid \operatorname{div}(z^\lambda X) = 0\} \tag{4.4}$$

are superconformal if $\lambda \notin \mathbb{Z}$. If one defines the standard holomorphic part of the volume element with constant coefficients as

$$\operatorname{dvol}(z,\theta_1,\ldots,\theta_n) = dz \otimes \frac{\partial}{\partial\theta_1} \otimes\cdots\otimes \frac{\partial}{\partial\theta_n}, \tag{4.5}$$

then

$$L_X(z^\lambda\cdot\operatorname{dvol}(z,\theta_1,\ldots,\theta_n)) = 0 \qquad \forall X \in \mathfrak{svect}^L_\lambda(1|n). \tag{4.6}$$

Thus one can view these as the algebras which preserve the z^λ-twisted holomorphic part of the standard volume element.

One easily verifies that $\mathfrak{svect}^L_\lambda(1|n) \cong \mathfrak{svect}^L_\mu(1|n)$ if and only if $\lambda-\mu \in \mathbb{Z}$. If $\lambda \in \mathbb{Z}$, the algebra $\mathfrak{svect}^L_\lambda(1|n)$ is not simple (so it is not superconformal according to Definition 4.1.2). It contains, however, a simple ideal $\mathfrak{svect}^{L'}_\lambda(1|n)$ of codimension $(1|0)$ if n is even, and of codimension $(0|1)$, if n is odd. This ideal is described by the exact sequence

$$0 \longrightarrow \mathfrak{svect}^{L'}_\lambda(1|n) \longrightarrow \mathfrak{svect}^L_\lambda(1|n) \longrightarrow f(z)\theta_1\cdots\theta_n\frac{\partial}{\partial z} \longrightarrow 0. \tag{4.7}$$

4.1.3 The algebras of contact vector fields and Möbius contact fields

These comprise the third and fourth series of superconformal algebras. They owe their names to the fact that they are supergeometric generalizations of the classical contact structures. Define the following one-forms on $\overline{\mathbb{C}^{1|n}} \setminus \{0\}$:

$$\alpha_n = dz + \sum_{i=1}^{n} \theta_i d\theta_i, \tag{4.8}$$

$$\alpha_n^M = dz + \sum_{i=1}^{n-1} \theta_i d\theta_i + z\theta_n d\theta_n. \tag{4.9}$$

The first one will be called a *contact form*, the second one a *Möbius contact form*. Then we define the algebra of contact vector fields on $\overline{\mathbb{C}^{1|n}} \setminus \{0\}$ as

$$\mathfrak{k}^L(1|n) := \left\{X \in \mathfrak{vect}^L(1|n) \mid L_X(\alpha_n) = f_X \cdot \alpha_n \text{ for some } f_X \in \mathcal{P}^L(n)\right\}, \tag{4.10}$$

and the algebra of Möbius contact vector fields as

$$\mathfrak{k}^M(1|n) := \left\{X \in \mathfrak{vect}^L(1|n) \mid L_X(\alpha_n^M) = f_X \cdot \alpha_n^M \text{ for some } f_X \in \mathcal{P}^L(n)\right\}. \tag{4.11}$$

These definitions mean that these vector fields preserve the forms (4.8) and (4.9) up to a factor, or equivalently, that they preserve their kernels. Therefore one often describes them as the algebras which preserve the Pfaff equations

$$\alpha_n(X) = 0 \qquad \text{for } X \in \mathfrak{vect}^L(1|n), \tag{4.12}$$

resp. $\alpha_n^M(X) = 0$. It is well known that in the case of $\mathfrak{k}^L(1|0) = \mathfrak{witt}$, i.e., in the absence of odd dimensions, the contact vector fields coincide with the conformal vector fields on $\mathbb{C}^\times$. There is no analog of the Möbius superalgebra for the non-super case.

All algebras $\mathfrak{k}^L(1|n)$ and $\mathfrak{k}^M(1|n)$, except for $\mathfrak{k}^L(1|4)$ and $\mathfrak{k}^M(1|5)$, are simple. The latter two contain a simple ideal of codimension $(1|0)$, denoted $\mathfrak{k}^{L\prime}(1|4)$, resp. of codimension $(0|1)$ denoted $\mathfrak{k}^{M\prime}(1|5)$. These two simple subalgebras are, in fact, two of the four exceptional superconformal algebras. The other two are $\mathfrak{m}^L(1)$, the algebra preserving the form $\beta = d\theta_1 + zd\theta_2 + \theta_2 dz$ on $\overline{\mathbb{C}^{1|2}} \setminus \{0\}$ up to a factor, and $\mathfrak{kas}^L \subset \mathfrak{k}^L(1|6)$, which is perhaps the only truly exceptional one, generated by certain polynomial functions on $\overline{\mathbb{C}^{1|6}} \setminus \{0\}$ (see [Shc99], [GLS05] for the definitions and [Le] for an extensive review).

For an explicit construction of these algebras, another description is more useful. First, note that $\mathcal{P}^L(N)$ can be endowed with a Lie superalgebra structure by introducing the *contact bracket* on it:

$$\{f, g\}_{kb} := (2 - E)(f)\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}(2 - E)(g) - \{f, g\}_{pb}. \tag{4.13}$$

Here, $E = \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i}$ is the so-called Euler operator, and $\{f, g\}_{pb}$ is the *Poisson bracket*:

$$\{f, g\}_{pb} := -(-1)^{p(f)} \sum_{i=1}^{N} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \tag{4.14}$$

In [Le], it is shown that $\mathcal{P}^L(N)$ with its contact bracket is isomorphic to $\mathfrak{k}^L(1|N)$. Specifically, we have:

Proposition 4.1.3. *To every $f \in \mathcal{P}^L(N)$, assign a vector field K_f by setting*

$$K_f := (2 - E)(f) \frac{\partial}{\partial z} - H_f + \frac{\partial f}{\partial z} E, \tag{4.15}$$

where

$$H_f := -(-1)^{p(f)} \sum_{i=1}^{N} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i} \tag{4.16}$$

assigns to each f a Hamiltonian vector field. Then

$$L_{K_f}(\alpha_N) = 2 \frac{\partial f}{\partial z} \alpha_N, \tag{4.17}$$

and

$$\mathfrak{k}^L(1|N) = \mathrm{Span}\{K_f \big| f \in \mathcal{P}^L(N)\}. \tag{4.18}$$

Furthermore,

$$[K_f, K_g] = K_{\{f,g\}_{kb}}. \tag{4.19}$$

We do not want to prove this statement, which can also be extended to spaces with more than one even coordinate, but refer to [GLS05], [Le] for an extensive analysis. For the Möbius algebras, an analogous description is also given in [GLS05], [Le].

4.1.4 Central extensions, critical dimensions

These infinitely many superconformal algebras all define superconformal surfaces of dimension $1|n$, but not all of these surfaces are suitable for superconformal field theories. A superconformal algebra can only be the algebra of a two-dimensional superconformal field theory if it possesses a nontrivial central extension. These extensions have been classified (for the algebras known at that time) in [KvdL89]. For a complete review, see [Le]. In general, the maximal dimension of a nontrivial central extension of a Lie algebra $\mathfrak{g}$ is the dimension of its second cohomology group $H^2(\mathfrak{g})$. To find suitable representatives for these cohomology classes can be a quite nontrivial job. We will only list the extensions here, for information on how to obtain them, and how to represent them by superfields, see [KvdL89], [Le], [Sac].

Of the contact algebras, the following ones have nontrivial central extensions: $\mathfrak{witt}$ has a one-dimensional extension, the Virasoro algebra $\mathfrak{vir}$. The algebras

$\mathfrak{k}^L(1|1)$, $\mathfrak{k}^L(1|2)$, $\mathfrak{k}^L(1|3)$ each have a one-dimensional extension, while $\mathfrak{k}^{L'}(1|4)$ has a 3-dimensional one. These extensions are called the Neveu-Schwarz algebras $\mathfrak{ns}(1)$, $\mathfrak{ns}(2)$, $\mathfrak{ns}(3)$ and $\mathfrak{ns}(4)$. The latter is one particular of the three extensions of $\mathfrak{k}^L(1|4)$. For $n \geq 5$, none of the $\mathfrak{k}^L(1|n)$ possesses an extension.

Of the Möbius contact algebras, each of the algebras $\mathfrak{k}^M(1|1)$, $\mathfrak{k}^M(1|2)$, $\mathfrak{k}^M(1|3)$, $\mathfrak{k}^M(1|4)$ has a one-dimensional extension. These extensions are dubbed the Ramond algebras $\mathfrak{r}(n)$. None of the $\mathfrak{k}^M(1|n)$ for $n \geq 5$ has a nontrivial extension.

Of the other superconformal algebras, $\mathfrak{vect}^L(1|1)$, $\mathfrak{vect}^L(1|2)$, $\mathfrak{svect}^L_\lambda(1|2)$ and $\mathfrak{m}^L(1)$ each have a unique nontrivial central extension.

This is the complete list, so altogether, there are at best 15 possibilites to define superconformal field theories if one adopts Definition 4.1.2 for a superconformal algebra. There might, however, be other possbilities if one allows loop or Kac-Moody algebras. Yet, not all of these 15 theories can describe the worldsheets of a superstring theory. In order to be able to define such a theory free of anomalies, there must exist a spacetime of dimension $D_{crit} \geq 0$ (obviously, one would prefer $D_{crit} \geq 4$), in which the (super-)conformal anomaly cancels [GSW87], [DP88]. This is quite a severe restriction. The Virasora algebra has $D_{crit} = 26$, $\mathfrak{ns}(1)$ and $\mathfrak{r}(1)$ both have $D_{crit} = 10$, and $\mathfrak{ns}(2)$ and $\mathfrak{r}(2)$ have $D_{crit} = 2$. For the other centrally extendable algebras, the critical dimension is ≤ 1 [LS07].

4.2 Spin surfaces and super Riemann surfaces

4.2.1 Complex supersurfaces

In all of the following work we will be concerned with orientable smooth supersurfaces of dimension $2|2$. Let us assume we are given such a surface $\mathcal{M}$ together with a complex structure, that is, we have a complex supermanifold $\mathcal{M}$ of dimension $1|1$. Let (z, θ) and (z', θ') be two overlapping complex coordinate charts on $\mathcal{M}$. Then the transition function between these must have the form

$$\begin{aligned} z' &= f(z) \\ \theta' &= g(z)\theta, \end{aligned} \tag{4.20}$$

because it must be a morphism of superalgebras on each stalk. Writing the structure sheaf as $\mathcal{O}_\mathcal{M} = \mathcal{O}_{\mathcal{M},\bar{0}} \oplus \mathcal{O}_{\mathcal{M},\bar{1}}$, we find that $\mathcal{O}_{\mathcal{M},\bar{0}}$ is simply the sheaf $\mathcal{O}_M$ of holomorphic functions on the underlying Riemann surface $M = \mathcal{M}_{red}$,[1] while $\mathcal{O}_{\mathcal{M},\bar{1}}$ is a sheaf of locally free modules of rank $0|1$ over $\mathcal{O}_{\mathcal{M},\bar{0}}$. Thus, setting

$$L := \Pi(\mathcal{O}_{\mathcal{M},\bar{1}}) \tag{4.21}$$

[1]In complex supergeometry, also even nilpotent elements can occur in the structure sheaf. In this case, $\mathcal{M}_{red}$ denotes the complex analytic space where one has only divided out the ideal of odd elements from the structure sheaf, while $\mathcal{M}_{rd}$ denotes the underlying space, i.e. the completely reduced one. In this work, we do not have to make a distinction, since no non-reduced ordinary complex spaces occur.

turns $\mathcal{M}$ into a Riemann surface endowed with a locally free sheaf of $\mathcal{O}_M$-modules of rank 1, i.e., a line bundle.

Conversely, it is obvious that starting with a pair (M, L) consisting of a Riemann surface and a holomorphic line bundle L, we can produce a $1|1$-dimensional complex supermanifold $\mathcal{M}$ by just setting $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}} = \mathcal{O}_M \oplus \Pi(\Gamma(L)))$, where $\Gamma(L)$ is the sheaf of holomorphic sections of L. Thus we have shown

Proposition 4.2.1. *There is a bijection between the set of pairs (M, L) of Riemann surfaces with a holomorphic line bundle and the set of complex supermanifolds $\mathcal{M}$ of dimension $1|1$.*

In fact, we will solely be concerned with compact (super) Riemann surfaces without boundary later on, but Prop. 4.2.1 holds also in the noncompact case.

This is the point where one has to clarify an important conceptual issue. In much of the literature (e.g., [CR88]), the transition functions between (z, θ) and (z', θ') are not given by (4.20), but rather by expressions like

$$\begin{aligned} z' &= f(z) + \psi(z)\theta \\ \theta' &= \eta(z) + g(z)\theta, \end{aligned} \tag{4.22}$$

where $\psi(z), \eta(z)$ are functions taking “odd values”. As remarked earlier, this point of view reflects the idea of functions taking values in some Grassmann algebra which we prefer to avoid in this work. Instead, we want to keep a clear distinction between the notion of a single supermanifold and that of a family of supermanifolds.

On a single ringed space $\mathcal{M}$, where each stalk is just an exterior algebra over the algebra of germs of holomorphic functions, it does not make any sense to speak of functions depending only on z as being odd. If, however, one looks at a family $\mathcal{T} \times \mathcal{M}$, where $\mathcal{T}$ is supermanifold with structure sheaf $\mathcal{O}_{\mathcal{T}}$, then the transition maps on $\mathcal{M}$ may contain odd germs from the stalk of the base $\mathcal{T}$. Then $\psi(z), \eta(z)$ can be understood as being proportional to such an odd germ. The geometric picture describing this situation most accurately is to think of the presence of these odd germs as deformations of the transition function (4.20) along odd dimensions of the base. Thus the transition function (4.20) describes the family $\mathcal{T} \times \mathcal{M}$ restricted to the base $\mathcal{T}_{red}$, while (4.22) describes the full family, which can be understood as a deformation involving odd and even parameters.

To illustrate this point, consider the following example. Let $\mathcal{M}$ be a complex $1|1$-dimensional supermanifold, and let $\mathcal{T} = \operatorname{Spec} \Lambda_1^{\mathbb{C}} = (\{*\}, \Lambda_1^{\mathbb{C}})$ be the complex superpoint with one odd dimension. Let τ be the generator of Λ_1. Now consider the family $\mathcal{T} \times \mathcal{M}$. Let $\mathcal{U}, \mathcal{V}$ be two superdomains on $\mathcal{M}$. The transition function between $\mathcal{U}$ and $\mathcal{V}$ is then a morphism of families: a map of superdomains parametrized by the base. This means it is a map $\Phi : \mathcal{T} \times \mathcal{U} \to \mathcal{T} \times \mathcal{V}$ which

makes the diagram

$$\begin{tikzcd} \mathcal{T} \times \mathcal{U} \arrow[rr, "\Phi"] \arrow[dr, "\Pi_{\mathcal{T}}"'] & & \mathcal{T} \times \mathcal{V} \arrow[dl, "\Pi_{\mathcal{T}}"] \\ & \mathcal{T} & \end{tikzcd} \tag{4.23}$$

commutative. This implies that Φ can be written as $\Pi_{\mathcal{T}} \times (\Phi_2 : \mathcal{T} \times \mathcal{U} \to \mathcal{V})$. Fix a point p in the underlying domain U which gets mapped to $\phi(p)$ by the underlying homeomorphism ϕ of Φ_2. The sheaf map φ associated with Φ_2 maps the stalk $\mathcal{O}_{\mathcal{V},\phi(p)}$ into $\mathcal{O}_{\mathcal{U},p} \otimes \Lambda_1^{\mathbb{C}}$. Let z, θ be the generators of the stalk $\mathcal{O}_{\mathcal{V},\phi(p)}$, then their image under φ is

$$z' = f(z) + \tau g(z)\theta \tag{4.24}$$
$$\theta' = h(z)\theta + \tau k(z) \tag{4.25}$$

where f, g, h, k are germs of holomorphic functions in the stalk $\mathcal{O}_{\mathcal{U},p}$. This is precisely the form of (4.22).

For smooth supermanifolds, it always suffices to study just families over the superpoints $\mathcal{P}(\Lambda_n)$, $n \in \mathbb{N}_0$. One may view this fact as the extension of Ehresmann's theorem [Ehr51] to the super case: there are no nontrivial deformations of a smooth structure by real parameters. For complex supermanifolds, however, even families over purely even complex base spaces can be nontrivial, just as in the case of classical complex geometry.

This discussion should make it clear that, although Prop. 4.2.1 states that the set of single complex $1|1$-dimensional supermanifolds (i.e., families over the point $\operatorname{Spec}\mathbb{C}$) is in one-to-one correspondence with pairs of Riemann surfaces and line bundles, the Teichmüller and moduli spaces of these two types of structures will be different. These spaces describe deformations of structures, and as seen above, there are more deformations of a super object than just the ones of the classical underlying object.

All supersurfaces of complex dimension $1|n$ are superconformal surfaces, namely they are $\mathfrak{vect}^L(1|n)$-surfaces. All other superconformal surfaces are subspecies of the $\mathfrak{vect}^L(1|n)$-surfaces, since all superconformal algebras can be realized as subalgebras of algebras of vector fields.

4.2.2 $\mathfrak{k}^L(1|1)$-surfaces and spin curves

The notion of a super Riemann surface (SRS) was introduced by Friedan [Fri86] in the context of superstring theory, which can be viewed as a special type of 2D supergravity. In some of the mathematical literature [Man91], [Man97], the term SUSY_1-curve is used for an SRS. It is defined as a complex $1|1$-dimensional supermanifold $\mathcal{M}$ with the additional property that it possesses a maximally nonintegrable distribution $\mathcal{D} \subset \mathcal{TM}$ of rank $0|1$. This means that the pairing given

by

$$\begin{array}{rcl} \mathcal{D} \otimes \mathcal{D} & \to & \mathcal{TM}/\mathcal{D} \\ X \otimes Y & \mapsto & [X,Y]/\mathcal{D}, \end{array} \tag{4.26}$$

where $[\cdot,\cdot]$ is the Lie bracket, is an isomorphism. Recalling that a distribution $\mathcal{D}$ is integrable if and only if $[\mathcal{D},\mathcal{D}] \subseteq \mathcal{D}$, it is obvious why a distribution with the above property is called maximally non-integrable. In ordinary geometry, such a distribution corresponds precisely to a contact structure, and we will show that in the super case, it is preserved by the contact algebra $\mathfrak{k}^L(1|1)$.

A theorem of LeBrun and Rothstein [LR88] states that, given a distribution $\mathcal{D}$ with properties (4.26), one can always find a local coordinate system (z,θ) such that $\mathcal{D}$ is locally generated by the odd vector field

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}. \tag{4.27}$$

That means that the transition functions of a super Riemann surface have to preserve D up to an invertible factor. Writing again

$$\begin{array}{rcl} z' & = & f(z) \\ \theta' & = & g(z)\theta, \end{array} \tag{4.28}$$

one deduces that

$$D = \theta f'(z)\frac{\partial}{\partial z'} + g(z)\frac{\partial}{\partial\theta'}. \tag{4.29}$$

So in order to have $D \sim D'$, we have to require

$$f'(z) = g(z)^2. \tag{4.30}$$

This differs from the result in [CR88] because we only look at a single SRS here, while the authors of [CR88] implicitly study a family of SRS parametrized by a supermanifold (cf. the discussion in the previous section). Equation (4.30) states that under super coordinate transformations, the odd coordinate θ transforms like a section of a spin bundle $S = K^{1/2}$. By an argument analogous to that in Section 4.2.1, we conclude

Proposition 4.2.2. *There exists a bijection between the set of super Riemann surfaces and the set of pairs (M,S), where M is Riemann surface and S is a spin bundle on M.*

Riemann surfaces with a spin structure are also called *spin curves.* For an extensive investigation, see, e.g., [Ati71]. On a Riemann surface, a spin bundle is simply a holomorphic line bundle which transforms by the square root of the transition functions of the canonical bundle $K = T^*M$. If the surface M has genus g, then there exist 2^{2g} spin structures on M, one for each element of $H^1(M,\mathbb{Z}_2)$. The reason is that to fix a square root of K uniquely, one has to choose the sign

for the square roots of the transition functions which one picks up running along any homologically nontrivial 1-cycle on M. Often a spin bundle is denoted as $K^{1/2}$, but that notation is, of course, not unique if the genus of the surface is greater than zero. The same goes for other half powers of K, like $K^{3/2}$. All this implies that a Riemann surface M is able to carry at least 2^{2g} non-equivalent SRS-structures. But again, by the same arguments as outlined in the previous section, it is the possibility of odd deformation parameters which makes the Teichmüller and moduli spaces for SRS interesting, even though the underlying spaces of these can at this point already be expected to be just the spin Teichmüller and spin moduli spaces.

Super Riemann surfaces are, if one uses the naming of a superconformal surface after the superconformal structure it carries, $\mathfrak{k}^L(1|1)$-surfaces. By definition, the algebra $\mathfrak{k}^L(1|1)$ preserves the contact form $\omega = dz + \theta d\theta$ up to a factor. This means it preserves its kernel. Using our convention (2.78) for the pairing between forms and vector fields, we see that

$$\langle f(z,\theta)\frac{\partial}{\partial z} + g(z,\theta)\frac{\partial}{\partial \theta}, dz + \theta d\theta\rangle = 0 \tag{4.31}$$

implies $f = g\theta$. Therefore the kernel of ω consists of elements of the form

$$g(z,\theta)\left(\theta\frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}\right), \tag{4.32}$$

and therefore coincides with the distribution $\mathcal{D}$ defined above.

4.2.3 Families of spin curves and of $\mathfrak{k}^L(1|1)$-curves

It is interesting to directly compare the situation for families of spin curves and SRS, because it shows how far the relation between these two types of objects reaches, as well as where the differences occur. We follow here the exposition given in [Man91], Chapter 2.

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a family of complex supermanifolds. The associated sheaf map $\pi^* : \mathcal{O}_\mathcal{B} \to \mathcal{O}_\mathcal{X}$ of this projection embeds the structure sheaf of the base into that of the total space. The relative tangent sheaf $\mathcal{T}_{\mathcal{X}/\mathcal{B}}$ is then defined as the subsheaf of vector fields in $\mathcal{T}_\mathcal{X}$ which annihilate $\pi^*(\mathcal{O}_\mathcal{B})$. One may think of them as those vector fields which point "vertically" along the fibers, such that the images of germs of functions from $\mathcal{O}_\mathcal{B}$ are treated as constants by these vector fields.

A family of super Riemann surfaces is a family $\pi : \mathcal{X} \to \mathcal{B}$ of complex supermanifolds such that the relative tangent sheaf contains a distribution $\mathcal{D} \subset \mathcal{T}_{\mathcal{X}/\mathcal{B}}$ of rank $0|1$ which is maximally non-integrable (compare with Definition 4.26). One may think of this as a distribution satisfying Definition 4.26 on *every fiber.* One must be warned, however, that the concept of "fibers" parametrized by a supermanifold differs from that of fibers over an ordinary manifold or topological space, since the supermanifold is not specified by its topological points. It parametrizes

the fibers both by even as well as odd parameters, and only the former ones can be thought of as a parametrization in the sense of families of topological spaces.

By the considerations of the previous section, it is clear how to construct a family of $\mathfrak{k}^L(1|1)$-curves over a purely even base B_0. One starts with a family $\pi_0 : X_0 \to B_0$ of relative dimension $1|0$, i.e., a family of ordinary complex 1-dimensional manifolds. Choosing a line bundle L on X_0 and declaring it odd turns every fiber into a complex supermanifold of dimension $1|1$. To obtain a family of $\mathfrak{k}^L(1|1)$-surfaces, we choose a line bundle L for which there exists an isomorphism

$$\alpha : L \otimes L \to \mathcal{T}^*_{X_0/B_0}, \tag{4.33}$$

which means nothing else than the requirement that L restricts to a square root of the canonical bundle of each fiber. Bundles with this property are called *theta characteristics*[2] of π_0. Denoting the sheaf of sections of L by $\Gamma(L)$, we obtain a family of particular complex supermanifolds by setting $\mathcal{X} = (X_0, \mathcal{O}_{X_0} \oplus \Pi(\Gamma(L)))$ and a projection $p : \mathcal{X} \to X_0$, as well as a projection $\pi = \pi_0 \circ p : \mathcal{X} \to B_0$. This family can explicitly be endowed with the structure of a family of super Riemann surfaces: take a local relative coordinate z on X_0 and a section θ of L such that $\alpha(\theta, \theta) = d_{X/B_0} z$ (d_{X/B_0} denotes the relative differential). Such a section always exists, because α is an isomorphism. Now we can set (z, θ) as a local relative coordinate system on $\mathcal{X}$, and define $D := \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$. Then the distribution $\mathcal{O}_X D \subset \mathcal{T}_{\mathcal{X}/B_0}$ satisfies the condition of maximal non-integrability, and we have turned X into a family of super Riemann surfaces. This construction does not depend on the relative coordinate z that one starts with: if we had started with $z' = f(z)$, then we would have ended up with $\theta' = \sqrt{\frac{\partial f}{\partial z}}\theta$, and thus with

$$D' = \left(\frac{\partial f}{\partial z}\right)^{-\frac{1}{2}} D. \tag{4.34}$$

Since f is conformal, i.e., holomorphic and with nowhere vanishing derivative, the coordinate system (z', θ') would thus have produced the same distribution in $\mathcal{T}_{\mathcal{X}}$. In [Man91] it is shown that the converse also holds: every family $\pi : \mathcal{X} \to \mathcal{B}$ of super Riemann surfaces reduces to a family $X_{red} \to B_{red}$ with a canonically determined theta characteristic.

Theorem 4.2.3. *Let a family $\pi_0 : X_0 \to B$ of complex supermanifolds of relative dimension $1|0$ be given. Then there exists a bijection between the following two sets:*

1. *relative theta characteristics of π_0 up to equivalence*

[2] Often the points of the half-period lattice on the Jacobian $\mathrm{Jac}(\Sigma)$ of a Riemann surface Σ are called the theta characteristics of Σ. This almost coincides with the above usage of the term: the divisor classes of line bundles which square to the canonical bundle are given by the half-period lattice shifted by the vector of Riemann constants. For more details see Section 8.2 or [GH94], [ACGH85].

2. $\mathfrak{k}^L(1|1)$*-families* $\pi : X \to B$ *such that* $X_{red} = X_{0,red}$ *for which* $\mathcal{O}_{X.\bar{0}} \cong \mathcal{O}_{X_0}$.

A universal family of super Riemann surfaces above their moduli space must therefore restrict to a family spin curves, as well. Indeed, the underlying spaces of the Teichmüller and moduli spaces of SRS turn out to be just the Teichmüller and moduli spaces of spin curves. The reference for the above Theorem and the preceding discussion is [Man91] and references therein, in particular [Del].

Chapter 5

The supermanifold $\mathcal{A}(\mathcal{M})$

In this chapter, we will construct a supermanifold $\mathcal{A}(\mathcal{M})$ of all almost complex structures on a given almost complex supermanifold $\mathcal{M}$. In subsequent Chapters we will identify the tangent spaces to the Teichmüller spaces that we wish to construct as certain subspaces of the tangent spaces of $\mathcal{A}(\mathcal{M})$. The Teichmüller spaces themselves can therefore be thought of as being glued together from local submanifolds of $\mathcal{A}(\mathcal{M})$.

5.1 Supermanifolds of sections of super vector bundles

Let $\pi : \mathcal{E} \to \mathcal{M}$ be a smooth super vector bundle over a compact supermanifold $\mathcal{M}$. We would like to give the set of sections

$$\Gamma(\mathcal{M}, \mathcal{E}) := \{\sigma : \mathcal{M} \to \mathcal{E} \,\big|\, \pi \circ \sigma = \mathrm{id}_{\mathcal{M}}\} \tag{5.1}$$

the "structure of a supermanifold". As always in the super context, to obtain this set is by far not sufficient for this purpose; it will only form the underlying space of such a supermanifold. The starting point of the construction is the inner Hom-object $\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{E})$ for the set $\mathrm{Hom}(\mathcal{M}, \mathcal{E}) = SC^{\infty}(\mathcal{M}, \mathcal{E})$. By definition, we have

$$\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{E})(\Lambda) = SC^{\infty}(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{E}). \tag{5.2}$$

This inner Hom-object is in general not a Banach supermanifold, for the same reason as in ordinary geometry: spaces of smooth maps usually only form Fréchet manifolds.

Next, define a functor $\hat{\Gamma}(\mathcal{M}, \mathcal{E}) : \mathsf{Gr} \to \mathsf{Sets}$ on the objects of Gr by setting

$$\hat{\Gamma}(\mathcal{M}, \mathcal{E})(\Lambda) := \Gamma(\mathcal{P}(\Lambda) \times \mathcal{M}, \Pi_{\mathcal{M}}^{*}\mathcal{E}). \tag{5.3}$$

Here, $\Pi_{\mathcal{M}}^{*}\mathcal{E}$ denotes the pullback of $\mathcal{E}$ along the projection $\Pi_{\mathcal{M}} : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$. For a morphism $\varphi : \Lambda \to \Lambda'$, we define

$$\begin{aligned} \hat{\Gamma}(\mathcal{M}, \mathcal{E})(\varphi) : \hat{\Gamma}(\mathcal{M}, \mathcal{E})(\Lambda) &\to \hat{\Gamma}(\mathcal{M}, \mathcal{E})(\Lambda') \\ \sigma &\mapsto \sigma \circ (\mathcal{P}(\varphi) \times \mathrm{id}_{\mathcal{M}}). \end{aligned} \tag{5.4}$$

The functor $\hat{\Gamma}(\mathcal{M},\mathcal{E})$ represents the global sections of the bundle $\pi : \mathcal{E} \to \mathcal{M}$ as a functor in $\mathsf{Sets}^{\mathsf{Gr}}$. As with $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{E})$, this functor cannot be a Banach supermanifold, but at most a Fréchet supermanifold. Nonetheless, it can be easily constructed, since it turns out that it is actually a superrepresentable $\overline{\mathbb{R}}$-supermodule.

Theorem 5.1.1. *Let $\pi : \mathcal{E} \to \mathcal{M}$ be a smooth super vector bundle. Then the functor $\hat{\Gamma}(\mathcal{M},\mathcal{E})$ is a superrepresentable $\overline{\mathbb{R}}$-module, i.e., there exists an $\mathbb{R}$-super vector space E such that*

$$\hat{\Gamma}(\mathcal{M},\mathcal{E}) \cong \overline{E}. \tag{5.5}$$

Proof. We first show that $\hat{\Gamma}(\mathcal{M},\mathcal{E})$ is indeed an $\overline{\mathbb{R}}$-module. Let $\sigma : \mathcal{P}(\Lambda)\times\mathcal{M} \to \mathcal{E}$ be a given section, and let $\lambda \in \Lambda_{\bar{0}}$ be an element of $\overline{\mathbb{R}}(\Lambda)$. Because we have $\mathcal{P}(\Lambda)(\Lambda') \cong \mathrm{Hom}(\Lambda,\Lambda')$ (cf. Prop. 2.1.13), every $p \in \mathcal{P}(\Lambda)(\Lambda')$ may be viewed as a morphism $p : \Lambda \to \Lambda'$. We then define the functor morphism

$$\lambda\cdot\sigma : \mathcal{P}(\Lambda)\times\mathcal{M} \to \mathcal{E} \tag{5.6}$$

componentwise:

$$\begin{aligned} (\lambda\cdot\sigma)_{\Lambda'} : \mathcal{P}(\Lambda)(\Lambda')\times\mathcal{M}(\Lambda') &\to \mathcal{E}(\Lambda') \\ (p,m) &\mapsto p(\lambda)\cdot\sigma_{\Lambda'}(p,m). \end{aligned} \tag{5.7}$$

Here, we have used the fact that $\sigma_{\Lambda'}(p,m) \in (\pi_{\Lambda'})^{-1}(m) \cong \mathcal{V}(\Lambda')$, i.e., that the fibres over all points $m \in \mathcal{M}(\Lambda')$ carry the structure of $\overline{\mathbb{R}}(\Lambda') = \Lambda'_{\bar{0}}$-modules.

Now consider the set of sections

$$E := \Gamma(\mathcal{M},\mathcal{E}\oplus\Pi\mathcal{E}) = \{\sigma : \mathcal{M} \to \mathcal{E}\oplus\Pi\mathcal{E} \,|\, \pi\circ\sigma = \mathrm{id}_{\mathcal{M}}\}. \tag{5.8}$$

This set carries the structure of an $\mathbb{R}$-super vector space: every section taking values in $\mathcal{E}$ is defined to be even, every one taking values in $\Pi\mathcal{E}$ to be odd. We claim that $\overline{E} \cong \hat{\Gamma}(\mathcal{M},\mathcal{E})$, i.e., that there exists a bijection between the sets $\overline{E}(\Lambda)$ and $\hat{\Gamma}(\mathcal{M},\mathcal{E})(\Lambda)$ for all $\Lambda \in \mathsf{Gr}$.

We will prove this locally, i.e., on every stalk, using the ringed space formalism. Denote the rank of the super vector bundle by $a|b$. Let $p \in M$ be a point in the underlying manifold $\mathcal{M}_{rd}$, and let $\sigma : \mathcal{P}(\Lambda_n)\times\mathcal{M} \to \mathcal{E}$ be an element of $\hat{\Gamma}(\mathcal{M},\mathcal{E})(\Lambda_n)$. Then σ is determined by a continuous map of the underlying spaces $s : \{*\}\times\mathcal{M}_{rd} \to \mathcal{E}_{rd}$, and by a sheaf map $\sigma^* : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{M}}\times\Lambda_n$. The sheaf map induces a stalk map between the stalk at $(\{*\},p)$ and the stalk at $s(p)$:

$$\begin{aligned} \sigma_p^* : \mathcal{O}_{\mathcal{E},s(p)} &\to \mathcal{O}_{\mathcal{M},p}\times\Lambda_n \\ f &\mapsto \sum_{I\subseteq\{1,\ldots,n\}} \tau_I\alpha_I(f), \end{aligned} \tag{5.9}$$

where each α_I is a homomorphism $\mathcal{O}_{\mathcal{E},s(p)} \to \mathcal{O}_{\mathcal{M},p}$ of superalgebras, and the sum runs over all increasingly ordered subsets. The $\tau_1,\ldots,\tau_n$ are the odd generators of Λ_n, and τ_I is the product of all the τ's indexed by I in the same order

(cf. Thm. 7.2.6 for this notation). By Thm. 7.2.6 we know that there exists a homomorphism $\alpha_0 : \mathcal{O}_{\mathcal{E},s(p)} \to \mathcal{O}_{\mathcal{M},p}$ of superalgebras, as well as 2^{n-1} odd and $2^{n-1}-1$ even derivations of $\mathcal{O}_{\mathcal{E},s(p)}$ such that the stalk map at p induced by any morphism $\sigma : \mathcal{P}(\Lambda_n) \times \mathcal{M} \to \mathcal{E}$ can be written as

$$\sigma_p^* = \alpha_0 \circ \exp\left(\sum_{I\subseteq\{1,\dots,n\}} \tau_I X_I\right). \tag{5.10}$$

Here, each X_I is a derivation of parity $|I|$ of $\mathcal{O}_{\mathcal{E},s(p)}$.

Now the fact that σ is a section implies that the composition with $\pi : \mathcal{E} \to \mathcal{M}$ must satisfy $\pi \circ \sigma = \Pi_{\mathcal{M}}$. That means that for any germ $f \in \mathcal{O}_{\mathcal{M},p}$, the composition of sheaf maps satisfies $\sigma^*\pi^*(f) = \Pi^*_{\mathcal{M}}(f)$. Inserting $\pi^*(f)$ into (5.10) yields immediately the conditions

$$\alpha_0(\pi^*(f)) = f, \qquad X_I(\pi^*(f)) = 0. \tag{5.11}$$

The first of these conditions means that the "underlying" section $\sigma : \mathcal{M} \to \mathcal{E}$, described by α_0, is a local inverse of the projection π, just as one would have expected. The second one tells us that all the derivations X_I in (5.10) have to be *relative*, i.e. they must annihilate the functions depending only on coordinates of the base.

Since a super vector bundle of rank $a|b$ is a locally free module of this rank over $\mathcal{O}_{\mathcal{M}}$, we know that the set of all homomorphisms α_0 is

$$\mathrm{Hom}_{SAlg}(\mathcal{O}_{\mathcal{E},s(p)}, \mathcal{O}_{\mathcal{M},p}) \cong (\mathcal{O}_{\mathcal{M},\bar{0},p})^a \oplus (\mathcal{O}_{\mathcal{M},\bar{1},p})^b. \tag{5.12}$$

The relative derivations $X : \mathcal{O}_{\mathcal{E},s(p)} \to \mathcal{O}_{\mathcal{E},s(p)}$ form a module of rank $a|b$ over $\mathcal{O}_{\mathcal{E},s(p)}$. Each of the X_I appears composed with α_0, which maps $\mathcal{O}_{\mathcal{E},s(p)}$ to $\mathcal{O}_{\mathcal{M},p}$, so we can consider each of the α_I in (5.9) actually as an element of an $\mathcal{O}_{\mathcal{M},p}$-module of rank $a|b$. So we can write

$$\alpha_I \in \begin{cases} (\mathcal{O}_{\mathcal{M},\bar{0},p})^a \oplus (\mathcal{O}_{\mathcal{M},\bar{1},p})^b & \text{if } |I| \text{ even} \\ (\mathcal{O}_{\mathcal{M},\bar{1},p})^a \oplus (\mathcal{O}_{\mathcal{M},\bar{0},p})^b & \text{if } |I| \text{ odd.} \end{cases} \tag{5.13}$$

Therefore, writing

$$V = V_{\bar{0}} \oplus V_{\bar{1}} = \left[(\mathcal{O}_{\mathcal{M},\bar{0},p})^a \oplus (\mathcal{O}_{\mathcal{M},\bar{1},p})^b\right] \oplus \left[(\mathcal{O}_{\mathcal{M},\bar{1},p})^a \oplus (\mathcal{O}_{\mathcal{M},\bar{0},p})^b\right] \tag{5.14}$$

shows us that we may identify the set of all stalk maps σ_p^* as in (5.9) with the vector space $\overline{V}(\Lambda_n)$. For $\Lambda_n = \mathbb{R}$, we thus obtain the space $V_{\bar{0}}$, which is the space (5.12). Since these are just the sections $\mathcal{M} \to \mathcal{E}$, we have $V_{\bar{0}} = E_{\bar{0}}$. For $\Lambda_n = \Lambda_1$, we obtain $V_{\bar{0}} \oplus \Pi(V_{\bar{0}})$ because $V_{\bar{1}} = \Pi(V_{\bar{0}})$, and this is isomorphic to $E_{\bar{0}} \oplus E_{\bar{1}}$. It is then clear that $\bar{E}(\Lambda) \cong \bar{V}(\Lambda)$ for all $\Lambda \in \mathsf{Gr}$, which completes the proof. □

Theorem 5.1.1 reflects the intuitive expectation that the sections of a super vector bundle on a super manifold should inherit the structure of a super vector space from the fibers, as it is the case for ordinary vector bundles. That the proof is somewhat involved shows, on the other hand, that things are considerably more complicated in the super setting. This is due to the fact that one does not have a collection of super vector spaces, parametrized by some topological space (where the sections would obviously form a super vector space even if they were not even continuous), but rather a collection parametrized by the smooth structure of the base manifold $\mathcal{M}$ *and* by the odd elements of the structure sheaf $\mathcal{O}_{\mathcal{M}}$. As a corollary, we obtain the following theorem of Molotkov [Mol84].

Corollary 5.1.2. *Let $\mathcal{M}$ be a supermanifold and $\mathcal{V}$ be a superrepresentable $\overline{\mathbb{R}}$-module. Then there exists an isomorphism of functors*

$$\widehat{SC}^{\infty}(\mathcal{M},\mathcal{V}) \cong \overline{SC^{\infty}(\mathcal{M},\mathcal{V}\oplus\Pi(\mathcal{V}))}, \tag{5.15}$$

which is natural in $\mathcal{M}$ and $\mathcal{V}$.

Proof. This follows directly from the application of Thm. 5.1.1 to the trivial super vector bundle $\Pi_{\mathcal{M}} : \mathcal{M}\times\mathcal{V}\to\mathcal{M}$. □

Actually, as in the classical case, the $\overline{\mathbb{R}}$-module $\hat{\Gamma}(\mathcal{M},\mathcal{E})$ of global sections of a trivial super vector bundle is a module over the globally defined superfunctions, but this fact will not be used in the following.

5.2 Construction of the supermanifold $\mathcal{A}(\mathcal{M})$

5.2.1 Almost complex structures and Q-structures

An almost complex structure on a supermanifold $\mathcal{M}$ is a section

$$J : \mathcal{M} \to \mathcal{E}nd(\mathcal{T}\mathcal{M})$$

of the endomorphism bundle of $\mathcal{T}\mathcal{M}$, such that $J^2 = -\mathbb{1}$. Here, $\mathbb{1}$ denotes the section of $\mathcal{E}nd(\mathcal{T}\mathcal{M})$ which is constantly the identity.

On a supermanifold of dimension $2n|2n$, there might be odd sections as well as even ones which satisfy $J^2 = -\mathbb{1}$. So in the case of a smooth $2|2$-dimensional supermanifold, the case we want to study in this work, there might exist odd almost complex structures. Although they are an interesting case in their own right, we will ignore them in this work. The main reason is that the geometry defined by them is quite different from the one defined by an even almost complex structure. The latter corresponds to a reduction of the frame group of a supermanifold from $GL(2n|2m;\mathbb{R})$ to $GL(n|m;\mathbb{C})$ and can therefore be viewed as a superization of the concept of an almost complex structure. An odd one, however, corresponds to a reduction of the frame group from $GL(2n|2n;\mathbb{R})$ to the supergroup $Q(n)$ (the so-called queer analogue of the general linear group [Leiar]), so

the term "Q-structure" seems more appropriate for them (see [Vai85], [Vai84] for this terminology and an in-depth discussion). Q-structures are rigid [Vai85]. The geometry defined them is completely different from the one of complex supermanifolds. In particular, it is non-supercommutative.

5.2.2 Construction of $\mathcal{A}(\mathcal{M})$

We denote from now on by $\mathcal{A}(\mathcal{M})(\Lambda)$ the almost complex structures in $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda)$, i.e. the sections

$$J : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{P}(\Lambda) \times \mathcal{E}nd(\mathcal{TM}) \tag{5.16}$$

such that $J^2 = -\mathbb{1}_\Lambda$. In view of the definition given above, these sections should more precisely be called almost complex structures on the trivial family over $\mathcal{P}(\Lambda)$ with fiber $\mathcal{M}$. As always, the higher Λ-points of a superspace of maps between two super objects $\mathcal{M}, \mathcal{N}$ are maps $\mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{N}$.

It will be shown that the almost complex structures on an almost complex supermanifold $\mathcal{M}$ form a submanifold in $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$. The strategy will be to extend the technique invented by U. Abresch and A.E. Fischer [Tro92] for the construction of holomorphic coordinates on the set of almost complex structures of an ordinary manifold. This will turn each of the sets $\mathcal{A}(\mathcal{M})(\Lambda)$ into a complex manifold, and it will subsequently be shown that the Abresch-Fischer charts which provide these manifold structures can be constructed in a functorial manner with respect to Λ. This in turn allows us to make them the points of supercharts on the functor $\mathcal{A}(\mathcal{M})$.

Lemma 5.2.1. *The points $\mathcal{A}(\mathcal{M})(\Lambda)$ form a subfunctor of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$ in $\mathsf{Sets}^{\mathsf{Gr}}$.*

Proof. Each set $\mathcal{A}(\mathcal{M})(\Lambda)$ is obviously a subset of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda)$. It remains to be shown that the family of inclusions $\{\mathcal{A}(\mathcal{M})(\Lambda) \subset \hat{\Gamma}(\mathcal{E}nd)(\Lambda) | \Lambda \in \mathsf{Gr}\}$ is a functor morphism. This means we have to prove that for a morphism $\varphi : \Lambda \to \Lambda'$, we can define a morphism

$$\mathcal{A}(\mathcal{M})(\varphi) : \mathcal{A}(\mathcal{M})(\Lambda) \to \mathcal{A}(\mathcal{M})(\Lambda') \tag{5.17}$$

in such a way that it is compatible with the image of φ under $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$. This image is the map (recall that $\mathcal{P}$ is contravariant)

$$\begin{aligned} \hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\varphi) : \hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda) &\to \hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda') \\ S &\mapsto S \circ (\mathcal{P}(\varphi) \times \mathrm{id}_{\mathcal{M}}). \end{aligned} \tag{5.18}$$

We will show that defining $\mathcal{A}(\mathcal{M})(\varphi)$ as the restriction

$$\mathcal{A}(\mathcal{M})(\varphi) = \hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\varphi)\big|_{\mathcal{A}(\mathcal{M})(\Lambda)} \tag{5.19}$$

does the job. $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$ is an $\overline{\mathbb{R}}$-algebra, therefore $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\varphi)$ is a map between a $\Lambda_{\bar{0}}$- and a $\Lambda'_{\bar{0}}$-algebra whose multiplications are the components of a

functor morphism. Therefore, if $\sigma \in \mathcal{A}(\mathcal{M})(\Lambda)$ is given and we denote for brevity $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\varphi)$ by φ and by μ the multiplication in $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$, we have a commutative square

$$\begin{tikzcd} (\sigma,\sigma) \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\mu_\Lambda"'] & (\varphi(\sigma),\varphi(\sigma)) \arrow[d, mapsto, "\mu_{\Lambda'}"] \\ -\mathbb{1}_\Lambda \arrow[r, mapsto, "\varphi"] & -\mathbb{1}_{\Lambda'} \end{tikzcd} \quad (5.20)$$

Thus, $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\varphi)$ preserves the property of a section to be of square $-\mathbb{1}$, and restricting it to $\mathcal{A}(\mathcal{M})(\Lambda)$ yields a well defined map $\mathcal{A}(\mathcal{M})(\varphi)$, turning the sets $\mathcal{A}(\mathcal{M})(\Lambda)$ into a subfunctor of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))$. □

We first show that we can turn each of the sets $\mathcal{A}(\mathcal{M})(\Lambda)$ into an ordinary manifold:

Theorem 5.2.2. *For each $\Lambda \in \mathsf{Gr}$, the set $\mathcal{A}(\mathcal{M})(\Lambda)$ is a submanifold of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda)$.*

Proof. $\mathcal{A}(\mathcal{M})(\Lambda)$ is a subset of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda)$. In the following, we keep some $J_0 \in \mathcal{A}(\mathcal{M})(\Lambda)$ fixed. If $\mathcal{A}(\mathcal{M})(\Lambda)$ is indeed a manifold, then its tangent space at J_0 will be

$$T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda) = \left\{ H \in \hat{\Gamma}(\mathcal{E}nd)(\Lambda) \middle| J_0 H + H J_0 = 0 \right\} \tag{5.21}$$

as one sees from formally differentiating $J^2 = -\mathbb{1}_\Lambda$. For brevity, we will write just $\mathbb{1}$ instead of $\mathbb{1}_\Lambda$ in this proof, since this is the only component of $\mathbb{1}$ that will appear.

Now let $0 \in U$ be a neighbourhood of zero in $T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda)$ such that $(\mathbb{1}+H)$ is invertible for all $H \in U$. Then define

$$\begin{aligned} \psi_{J_0} : U &\rightarrow \mathcal{A}(\mathcal{M})(\Lambda) \\ H &\mapsto (\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} =: J. \end{aligned} \tag{5.22}$$

Clearly, $J^2 = -\mathbb{1}$ if and only if $J_0^2 = -\mathbb{1}$, so the range of ψ is in $\mathcal{A}(\mathcal{M})(\Lambda)$. Solving for H, one finds the inverse of ψ to be

$$\psi_{J_0}^{-1}(J) = (J - J_0)(J + J_0)^{-1} \tag{5.23}$$

This already proves that ψ provides a chart around J_0, because we have constructed a bijective map from a neighbourhood around J_0 in $\mathcal{A}(\mathcal{M})(\Lambda)$ to the linear space $T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda)$. Such charts are available around any $J_0 \in \mathcal{A}(\mathcal{M})(\Lambda)$. We have to show that the overlaps are given by smooth maps. Therefore let J be in the intersection of two charts, i.e.,

$$(\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} = J = (\mathbb{1}+K)J_1(\mathbb{1}+K)^{-1}, \tag{5.24}$$

with $K \in T_{J_1}\mathcal{A}(\mathcal{M})(\Lambda)$. Using (5.23), we find

$$K = \left[(\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} - J_1\right]\left[(\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} + J_1\right], \tag{5.25}$$

which is, clearly, smooth with respect to H whenever $(\mathbb{1}+H)$ is invertible.

To see that $\mathcal{A}(\mathcal{M})(\Lambda)$ is actually a submanifold of $\hat{\Gamma}(\mathcal{E}nd)(\Lambda)$, we must show that there exists a tubular neighbourhood of $\mathcal{A}(\mathcal{M})(\Lambda)$ around $J_0 \in \hat{\Gamma}(\mathcal{E}nd)(\Lambda)$. This means we must find a chart ϕ for $V \subset \hat{\Gamma}(\mathcal{E}nd)(\Lambda)$ around J_0, which locally flattens $\mathcal{A}(\mathcal{M})(\Lambda)$, i.e., such a chart $\phi(V)$ admits a direct sum decomposition into a subspace which provides the chart for $\mathcal{A}(\mathcal{M})(\Lambda) \cap V$ and a complement. For this, note that every $H \in \hat{\Gamma}(\mathcal{E}nd(\mathcal{TM}))(\Lambda)$ may be decomposed as $H = H^+ + H^-$, with

$$H^+ = \frac{1}{2}(H + J_0 H J_0), \qquad H^- = \frac{1}{2}(H - J_0 H J_0)$$

such that $J_0 H^+ + H^+ J_0 = 0$, i.e., an anticommuting part with respect to J_0, and $J_0 H^- - H^- J_0 = 0$, i.e., a commuting part. This splits

$$\hat{\Gamma}(\mathcal{E}nd)(\Lambda) = T^+_{J_0} \oplus T^-_{J_0} \tag{5.26}$$

where $T^+_{J_0}$ denotes the space of endomorphisms anticommuting with J_0, and $T^-_{J_0}$ denotes the space of those commuting with J_0. Now take a neighbourhood $V \subset \hat{\Gamma}(\mathcal{E}nd)(\Lambda)$ such that $V \cap \mathcal{A}(\mathcal{M})(\Lambda) = U$. Then $\mathrm{id} : V \to V$ is a chart around J_0 for $\hat{\Gamma}(\mathcal{E}nd)(\Lambda)$, and $\psi^{-1}_{J_0} : U \to T^+_{J_0}$ is a chart for $\mathcal{A}(\mathcal{M})(\Lambda) \cap V$ whose image lies in a direct subspace of V. It remains to check that there exists a neighbourhood around J_0 in $T^+_{J_0}$ which does not contain any points of $\mathcal{A}(\mathcal{M})(\Lambda)$. Indeed, consider an $H \in T^+_{J_0}$ and check the equation

$$(J_0 + H)^2 = J_0^2 + HJ_0 + J_0 H + H^2 = -\mathbb{1}. \tag{5.27}$$

Since H commutes with J_0, this is equivalent to

$$H(2J_0 + H) = 0.$$

We are not interested in the $H = 0$ solution, so we look for solutions of $2J_0 + H = 0$. It is clear that there exists a neighbourhood of $0 \in T^+_{J_0}$ for which this equation has no solutions, because J_0 is invertible. This shows that $\mathcal{A}(\mathcal{M})(\Lambda)$ is indeed a submanifold of $\hat{\Gamma}(\mathcal{E}nd)(\Lambda)$. □

In fact, we get even more, namely complex coordinates on $\mathcal{A}(\mathcal{M})(\Lambda)$. The construction in the following proof is an adaptation of the idea developed in [Tro92] for ordinary almost complex manifolds.

Theorem 5.2.3. $\mathcal{A}(\mathcal{M})(\Lambda)$ *is a complex manifold.*

Proof. This will be shown by directly constructing holomorphic coordinates. First, observe that $\mathcal{A}(\mathcal{M})(\Lambda)$ is already an almost complex manifold: for any $J \in \mathcal{A}(\mathcal{M})(\Lambda)$, the tensor J itself provides the almost complex structure on its tangent space:

$$\begin{aligned} \Phi : T_J\mathcal{A}(\mathcal{M})(\Lambda) &\to T_J\mathcal{A}(\mathcal{M})(\Lambda) \\ H &\mapsto JH \end{aligned} \tag{5.28}$$

This is a well defined map, because for each $H \in T_J\mathcal{A}(\mathcal{M})(\Lambda)$, we have

$$J(JH) + (JH)J = -H + (JH)J = -H - HJJ = 0,$$

so the image is again in $T_J\mathcal{A}(\mathcal{M})(\Lambda)$. Now we claim that

$$\psi_{J_0}^*(\Phi) = \Phi_{J_0}, \tag{5.29}$$

where and ψ_{J_0} is the chart defined in (5.22) and Φ_{J_0} denotes the fixed almost complex structure on the vector space $T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda)$. If (5.29) holds true, then ψ_{J_0} is actually a holomorphic coordinate chart around J_0 for the almost complex structure Φ (5.28). Such a chart can be found around any J_0, and on two overlapping charts, we have the transition function (5.25), which is clearly holomorphic in H for H near zero. Therefore the statement (5.29) is equivalent to the assertion of the theorem.

Take an arbitrary $K \in T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda)$. We have to show that

$$D\psi_{J_0}(H)^{-1}\Phi_J D\psi_{J_0}(H)K = \Phi_{J_0}(K) = J_0K$$

where $H \in T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda)$ is mapped to J under ψ_{J_0}.

The map $G \mapsto G^{-1}$, defined on the invertible operators, has the derivative $K \mapsto -G^{-1}KG^{-1}$. So

$$\begin{aligned} D\psi_{J_0}(H)K &= KJ_0(\mathbb{1}+H)^{-1} - (\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1}K(\mathbb{1}+H)^{-1} \\ &= KJ_0(\mathbb{1}+H)^{-1} - JK(\mathbb{1}+H)^{-1} \end{aligned}$$

and

$$JD\psi_{J_0}(H)K = -J(J+J_0)K(\mathbb{1}+H)^{-1}$$

To find $D\psi_{J_0}(H)^{-1}$, we calculate the derivative of $\psi_{J_0}^{-1}$ from (5.23):

$$\begin{aligned} D\psi_{J_0}^{-1}(J)L &= L(J+J_0)^{-1} - (J-J_0)(J+J_0)^{-1}L(J+J_0)^{-1} \\ &= L(J+J_0)^{-1} - HL(J+J_0)^{-1} \\ &= (\mathbb{1}-H)L(J+J_0)^{-1} \end{aligned}$$

So, altogether, one has

$$\begin{aligned} D\psi_{J_0}(H)^{-1}\Phi_J D\psi_{J_0}(H)K &= D\psi_{J_0}(H)^{-1}JD\psi_{J_0}(H)K \\ &= (\mathbb{1}-H)(-J(J+J_0)K(\mathbb{1}+H)^{-1})(J+J_0)^{-1} \\ &= -(\mathbb{1}-H)\,(J(J+J_0)K))\,((J+J_0)(\mathbb{1}+H))^{-1} \\ &= -(\mathbb{1}-H)\,\big((\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1}(J+J_0)K\big)\cdot \\ &\quad \cdot((J+J_0)(\mathbb{1}+H))^{-1} \end{aligned} \tag{5.30}$$

Now use

$$(\mathbb{1}-H)(\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} = J_0(\mathbb{1}-H)$$

to obtain for (5.30)

$$-J_0(\mathbb{1}-H)(J+J_0)K((J+J_0)(\mathbb{1}+H))^{-1} \tag{5.31}$$

However

$$\begin{aligned}(\mathbb{1}-H)(J+J_0) &= (\mathbb{1}-(J-J_0)(J+J_0)^{-1})(J+J_0)\\ &= ((J+J_0)-(J-J_0))\\ &= 2J_0\end{aligned}$$

and additionally,

$$(J+J_0)(\mathbb{1}+H) = (J+J_0)\left(\mathbb{1}+(J-J_0)(J+J_0)^{-1}\right)$$

which, together with

$$(J+J_0)(J-J_0) = -(J-J_0)(J+J_0)$$

yields

$$(J+J_0)(\mathbb{1}+H) = ((J+J_0)-(J-J_0)) = 2J_0$$

But $J_0^{-1} = -J_0$, so

$$((J+J_0)(\mathbb{1}+H))^{-1} = -\frac{1}{2}J_0.$$

Inserting everything into (5.31), one obtains finally

$$\begin{aligned}D\psi(H)^{-1}\Phi_J D\psi(H)K &= -J_0(2J_0)K(-\frac{1}{2}J_0)\\ &= -KJ_0\\ &= J_0K,\end{aligned}$$

which was to be shown. □

In order to show that these complex manifold structures on the sets $\mathcal{A}(\mathcal{M})(\Lambda)$ assemble to form a complex supermanifold $\mathcal{A}(\mathcal{M})$, we have to show that the construction of Abresch-Fischer charts carried out in the above proofs can be performed in a functorial manner. Since every open subfunctor of a functor $\mathcal{F} \in \mathsf{Top}^{\mathsf{Gr}}$ which is locally isomorphic to superdomains is a restriction (Prop. 3.5.8), this means that around every $J_0 \in \mathcal{A}(\mathcal{M})(\Lambda)$, we have to find an open domain U such that $\mathcal{A}(\mathcal{M})\big|_U$ is isomorphic to an open subfunctor of a superrepresentable $\overline{\mathbb{C}}$-module. In order to achieve this, the following two lemmas will be of use.

Lemma 5.2.4. *Let $\overline{V}$ be a superrepresentable $\overline{\mathbb{K}}$-module in $\mathsf{Sets}^{\mathsf{Gr}}$. Let S be a subset of its underlying points: $S \subset \overline{V}(\mathbb{K})$. Then $\overline{V}(\epsilon_\Lambda)^{-1}(S) \subset \overline{V}(\Lambda)$ has the form $S + \overline{V}^{nil}(\Lambda)$. Here, the (not superrepresentable) $\mathbb{K}$-module $\overline{V}^{nil}$ is defined by (3.59).*

Proof. Since $\overline{V}$ is superrepresentable, there exists a super vector space V (a $\mathbb{K}$-supermodule in Sets), such that

$$\overline{V}(\Lambda) = (\Lambda \otimes V)_{\bar{0}} \qquad \text{and} \qquad \overline{V}(\varphi) = \varphi \otimes \mathrm{id}_V$$

for any morphism $\varphi : \Lambda \to \Lambda'$. For $S \subset \overline{V}(\mathbb{K})$, this means simply that $S \subset V_{\bar{0}}$, and additionally

$$\overline{V}(\Lambda) = (\Lambda \otimes V)_{\bar{0}} = \Lambda_{\bar{0}} \otimes V_{\bar{0}} + \Lambda_{\bar{1}} \otimes V_{\bar{1}}.$$

The terminal morphism $\epsilon_\Lambda : \Lambda \to \mathbb{K}$ maps all odd generators of Λ to zero. Therefore, the general preimage of S under ϵ_Λ contains $S = (\Lambda_{\bar{0}} \cap \mathbb{K}) \otimes S$, plus an arbitrary term proportional to at least one odd generator of Λ. These are $\Lambda_{\bar{1}} \otimes V_{\bar{1}}$ and $(\Lambda_{\bar{0}} \cap \Lambda^{nil}) \otimes V_{\bar{0}}$. The sum of the latter two sets forms $(\Lambda^{nil} \otimes V)_{\bar{0}} = \overline{V}^{nil}(\Lambda)$. □

Lemma 5.2.5. *Let $\mathcal{B}$ be an associative $\overline{\mathbb{K}}$-algebra with unit in $\mathsf{Sets}^{\mathsf{Gr}}$. Let $a \in \mathcal{B}(\mathbb{K})$ be an invertible elment of its underlying algebra, i.e. there exists $a^{-1} \in \mathcal{B}(\mathbb{K})$ such that $aa^{-1} = a^{-1}a = 1$ where 1 is the unit in $\mathcal{B}(\mathbb{K})$. Then $\mathcal{B}(\epsilon_\Lambda)^{-1}(a)$ is a set of invertible elements in $\mathcal{B}(\Lambda)$ for all $\Lambda \in \mathsf{Gr}$.*

Proof. Due to Corollary 3.4.3, every associative $\overline{\mathbb{R}}$-algebra with unit in $\mathsf{Sets}^{\mathsf{Gr}}$ arises from an ordinary associative super algebra B, that is, a super vector space with a bilinear morphism

$$\mu : B \times B \to B$$

Then $\mathcal{B}(\Lambda) = (\Lambda \otimes B)_{\bar{0}}$, and the algebra operation becomes a functor morphism $\bar{\mu} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, given pointwise by a map

$$\bar{\mu}_\Lambda : \mathcal{B}(\Lambda) \times \mathcal{B}(\Lambda) \to \mathcal{B}(\Lambda)$$

This map is uniquely given by the requirements that it be $\Lambda_{\bar{0}}$-linear and that it fulfill

$$\bar{\mu}_\Lambda(\lambda_1 \otimes a_1, \lambda_2 \otimes a_2) = \lambda_2\lambda_1 \otimes \mu(a_1, a_2).$$

Now by Lemma 5.2.4, the preimage of $a \in \mathcal{B}(\mathbb{K})$ under ϵ_Λ is a set of the form $a + \mathcal{B}^{nil}(\Lambda)$. That means any element of it can be written as $a+c$, where c contains nilpotent elements of Λ, which means that $\bar{\mu}_\Lambda(c, c) = 0$. Then $a+c$ has the inverse $a^{-1} - a^{-1}ca^{-1}$:

$$\begin{aligned}
\bar{\mu}_\Lambda(a + c, a^{-1} - a^{-1}ca^{-1}) &= \bar{\mu}_\Lambda(a, a^{-1}) + \bar{\mu}_\Lambda(a, -a^{-1}ca^{-1}) + \\
&\quad \bar{\mu}_\Lambda(c, a^{-1}) + \bar{\mu}_\Lambda(c, -a^{-1}ca^{-1}) \\
&= 1 - \bar{\mu}_\Lambda(1, ca^{-1}) + \bar{\mu}_\Lambda(c, a^{-1}) - \bar{\mu}_\Lambda(c, a^{-1}ca^{-1}) \\
&= 1
\end{aligned}$$

We have used the fact that $\bar{\mu}_\Lambda$ is $\Lambda_{\bar{0}}$-linear, as well as associativity. The last term vanishes because c appears twice and thus introduces the same nilpotent elements twice. □

Lemma 5.2.5 is the categorified version of the fact that in an algebra which contains nilpotent elements, an element is invertible if and only if its reduced element is. Lemma 5.2.4 shows that for a superrepresentable $\overline{\mathbb{K}}$-module $\mathcal{V}$, it makes sense to regard an element $v \in \mathcal{V}(\mathbb{K})$ of its underlying space also as an element of all higher sets of points $\mathcal{V}(\Lambda)$. To construct a chart on $\mathcal{A}(\mathcal{M})$, we first specify a $\overline{\mathbb{C}}$-module which the chart will map to.

Proposition 5.2.6. *Fix a $J_0 \in \mathcal{A}(\mathcal{M})(\mathbb{R})$. We define a functor $\hat{T}_{J_0}$ in $\mathsf{Sets}^{\mathsf{Gr}}$ by*

$$\begin{aligned} \hat{T}_{J_0}(\Lambda) &= T_{J_0}\mathcal{A}(\Lambda) && (5.32) \\ \hat{T}_{J_0}(\varphi) &= \hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\varphi)\big|_{\hat{T}_{J_0}(\Lambda)} && (5.33) \end{aligned}$$

for any morphism $\varphi : \Lambda \to \Lambda'$. Then $\hat{T}_{J_0}$ is a superrepresentable $\overline{\mathbb{C}}$-module.

Proof. We know that $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))$ is superrepresentable, so there exists associative superalgebra $(\mathcal{E}, \mu)$ such that $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M})) \cong \overline{\mathcal{E}}$ and such that the multiplication μ becomes the composition of endomorphisms. Since J_0 is an element of the underlying space of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))$ it is an element of $\mathcal{E}_{\bar{0}}$. We may split the algebra $\mathcal{E}$ into a direct sum of linear subspaces $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ by setting

$$\begin{aligned} \mathcal{E}_+ &:= \{a \in \mathcal{E} \mid aJ_0 + J_0 a = 0\} \\ \mathcal{E}_- &:= \{a \in \mathcal{E} \mid aJ_0 - J_0 a = 0\} \end{aligned}$$

This decomposition is direct, since any a may be written uniquely as

$$a = \frac{1}{2}(a - J_0 a J_0) + \frac{1}{2}(a + J_0 a J_0)$$

with the first summand in $\mathcal{E}_-$ and the second one in $\mathcal{E}_+$ and $\mathcal{E}_+ \cap \mathcal{E}_- = \{0\}$. We want to show that $\hat{T}_{J_0} \cong \overline{\mathcal{E}_+}$. Let $H = \lambda \otimes h$ be an element of $\overline{\mathcal{E}_+}(\Lambda)$, i.e., we have $\mu(J_0, h) = -\mu(h, J_0)$. In $\overline{\mathcal{E}}$, μ is replaced by $\overline{\mu}$ and we find

$$\overline{\mu}_\Lambda(J_0, H) = \lambda \otimes \mu(J_0, h) = -\lambda \otimes (h, J_0) = -\overline{\mu}(H, J_0).$$

Therefore one has $\hat{T}_{J_0}(\Lambda) = \overline{\mathcal{E}_+}(\Lambda)$.

Finally, the functors $\hat{T}_{J_0}$ and $\overline{\mathcal{E}_+}$ are well-defined because μ is a functor morphism. This means that for every morphism $\varphi : \Lambda \to \Lambda'$ one has a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(\Lambda) \times \mathcal{E}(\Lambda) & \xrightarrow{\mu_\Lambda} & \mathcal{E}(\Lambda) \\ {\scriptstyle \mathcal{E}(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{E}(\varphi)} \\ \mathcal{E}(\Lambda') & \xrightarrow{\mu_{\Lambda'}} & \mathcal{E}(\Lambda') \end{array} \quad . \qquad (5.34)$$

The property of an element in $\mathcal{E}(\Lambda)$ to anticommute with J_0 is therefore preserved under images $\mathcal{E}(\varphi)$ of morphisms $\varphi \in \mathsf{Gr}$. □

The $\overline{\mathbb{C}}$-supermodule $\hat{T}_{J_0}$ is the functor that represents the tangent space $T_{J_0}\mathcal{A}(\mathcal{M})$. Since each of the Abresch-Fischer charts around J_0 maps precisely this tangent space onto $\mathcal{A}(\mathcal{M})$, we have to prove that $\mathcal{A}(\mathcal{M})$ is locally (around J_0) isomorphic to $T_{J_0}\mathcal{A}(\mathcal{M})$.

Theorem 5.2.7. *Let J_0 be a fixed element of $\mathcal{A}(\mathcal{M})(\mathbb{R})$. Then there exists an open subfunctor $\mathcal{V}$ around J_0 in $\mathcal{A}(\mathcal{M})$ which is isomorphic to an open subfunctor around 0 in $\hat{T}_{J_0}$.*

Proof. First, we construct a coordinate neighbourhood of J_0 in $\mathcal{A}(\mathcal{M})(\mathbb{R})$. According to Thm. 5.2.2, such a local coordinate system is given by a neighbourhood U of zero in $\hat{T}(\mathbb{R})$ via the map

$$\begin{aligned} \psi_{J_0} : U &\rightarrow \mathcal{A}(\mathcal{M})(\mathbb{R}) \\ H &\mapsto (\mathbb{1}+H)J_0(\mathbb{1}+H)^{-1} =: J. \end{aligned} \tag{5.35}$$

We will refer to this map and its analog for the higher Λ_n-points simply as ψ. One observes that U is only limited by the requirement that $(\mathbb{1}+H)$ be invertible. In the following we will keep U fixed, and we denote the image of U under ψ as $V \subset \mathcal{A}(\mathcal{M})(\mathbb{R})$.

The claim that we want to prove states that there exists an isomorphism

$$\mathcal{A}(\mathcal{M})(\epsilon_{\Lambda_n})^{-1}(V) \cong \hat{T}_{J_0}(\epsilon_{\Lambda_n})^{-1}(U) \tag{5.36}$$

which provides an Abresch-Fischer chart on each point set $\mathcal{A}(\mathcal{M})(\epsilon_{\Lambda_n})^{-1}(V)$.

First, we characterise the set $(\mathcal{A}(\mathcal{M})(\epsilon_\Lambda))^{-1}(J_0)$. According to Lemma 5.2.4, it is a set of the form $J_0+\mathcal{A}^{nil}_{J_0}$, where $\mathcal{A}^{nil}_{J_0}$ are elements from $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\Lambda)$ which are proportional to nilpotent elements of Λ. So any element $\tilde{J}_0 \in (\mathcal{A}(\mathcal{M})(\epsilon_\Lambda))^{-1}(J_0)$ has the form $J_0 + J_0^{nil}$, and since

$$\tilde{J}_0^2 = (J_0 + J_0^{nil})^2 = -\mathbb{1} + J_0 J_0^{nil} + J_0^{nil} J_0 \tag{5.37}$$

we see that J_0^{nil} must anticommute with J_0. So,

$$\mathcal{A}^{nil}_{J_0}(\Lambda) = \{H \in \hat{\Gamma}(\mathcal{E}nd)^{nil}(\Lambda) \mid HJ_0 + J_0 H = 0\}, \tag{5.38}$$

implying that the points of $\mathcal{A}(\mathcal{M})$ "above" J_0 (i.e. those whose reduced element is J_0) are those elements of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\Lambda)$ which anticommute with J_0 and which are purely nilpotent. These are nothing else than $(T_{J_0}\mathcal{A}(\mathcal{M})(\Lambda))^{nil}$.

Fix an arbitrary $\tilde{J}_0 \in (\mathcal{A}(\mathcal{M})(\epsilon_\Lambda))^{-1}(J_0)$. It must be of the form $\tilde{J}_0 = J_0 + J_0^{nil}$. If it can be covered by our proposed coordinate map ψ, it must be possible to write it as

$$\tilde{J}_0 = (\mathbb{1}+K)J_0(\mathbb{1}+K)^{-1} \tag{5.39}$$

for some $K \in \hat{T}_{J_0}(\Lambda)$. Since $\epsilon_\Lambda : \hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\Lambda) \rightarrow \hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\mathbb{R})$ is a homomorphism of associative algebras, the above equation implies that

$$J_0 = \epsilon_\Lambda(\mathbb{1}+K)J_0\epsilon_\Lambda(\mathbb{1}+K)^{-1}$$

So $\epsilon_\Lambda(K) = 0$, i.e. K is proportional to nilpotent elements. Then $(\mathbb{1}+K)^{-1} = \mathbb{1} - K$, and setting $K = -\frac{1}{2}J_0 J_0^{nil}$, we find the desired element of $\hat{T}_{J_0}(\Lambda)$, whose image under ψ is $\tilde{J}_0$. The invertibility of $\mathbb{1}+K$ is assured by Lemma 5.2.5.

Now consider another $J \in V$, and an arbitrary $\tilde{J} \in (\mathcal{A}(\mathcal{M})(\epsilon_\Lambda))^{-1}(J)$. We again want to find $K \in \hat{T}_{J_0}(\Lambda)$, such that

$$\tilde{J} = (\mathbb{1}+K)J_0(\mathbb{1}+K)^{-1}.$$

Applying ϵ_Λ, we see that this implies

$$J = \epsilon_\Lambda(\mathbb{1}+K)J_0\epsilon_\Lambda(\mathbb{1}+K)^{-1}$$

which means that $\epsilon_\Lambda(K) = H$, where H is the element of $T_{J_0}\mathcal{A}(\mathbb{R})$ whose image under ψ is J. So $K = H + K^{nil}$, where K^{nil} is some nilpotent term. Applying the same reasoning to $\tilde{J}$ as for $\tilde{J}_0$, we find $K \in \hat{T}(\Lambda)$ such that $\tilde{J}$ is the image of K under ψ and the underlying endomorphism of K is $H \in U$. Here again, Lemma 5.2.5 assures the existence of $(\mathbb{1}+K)^{-1}$.

We can now conclude that any $J \in \mathcal{A}(\mathcal{M})(\epsilon_\Lambda)^{-1}(V)$ can be written as

$$J = (\mathbb{1}+K)J_0(\mathbb{1}+K)^{-1} \tag{5.40}$$

with $K \in \hat{T}_{J_0}(\epsilon_\Lambda)^{-1}(U)$. Conversely, inserting any such K into (5.40) will produce a J above V. This proves the claim. □

The following Theorem finishes the construction of $\mathcal{A}(\mathcal{M})$ as a complex supermanifold.

Theorem 5.2.8. *The Abresch-Fisher coordinate charts around each $J_0 \in \mathcal{A}(\mathcal{M})(\Lambda)$ provide holomorphic complex coordinates for $\mathcal{A}(\mathcal{M})$. Therefore $\mathcal{A}(\mathcal{M})$ is a complex supermanifold.*

Proof. It was shown that $\hat{T}_{J_0}$ is a superrepresentable $\overline{\mathbb{C}}$-module. Therefore, the Abresch-Fischer charts identify $\mathcal{A}(\mathcal{M})$ locally with a linear complex supermanifold. The compatibility of the charts is ensured by Thm. 5.2.3: the almost complex structure induced by any chart coincides with the globally defined almost complex structure $\Phi_{J_0} : T_{J_0}\mathcal{A}(\mathcal{M}) \to T_{J_0}\mathcal{A}(\mathcal{M})$. This makes all transition functions automatically holomorphic. □

The equations (5.37) and (5.38) above give us an important piece of information about the structure of $\mathcal{A}(\mathcal{M})$: in its odd directions, it coincides with its tangent space. Of course, every supermanifold can be thought of as being "linear" in its odd directions, as was already argued above. But our particular construction of a chart for $\mathcal{A}(\mathcal{M})$ as an open domain in a tangent space tells us that we can write every $J \in \mathcal{A}(\mathcal{M})(\Lambda)$ as a sum of a $J_0 \in \mathcal{A}(\mathcal{M})(\mathbb{R})$ and a nilpotent tangent vector, which is a consequence of the fact that the tangent space at J is defined by an algebraic property.

Corollary 5.2.9. *Let J be an element of $\mathcal{A}(\mathcal{M})(\mathbb{R})$ and let*

$$U_J(\Lambda) = \mathcal{A}(\mathcal{M})(\epsilon_\Lambda)^{-1}(J) \tag{5.41}$$

be the set of all elements of $\mathcal{A}(\mathcal{M})(\Lambda)$ which reduce to J under the morphism $\epsilon_\Lambda : \Lambda \to \mathbb{R}$. Then each $J' \in U_J$ can be written as

$$J' = J + H^{nil}, \tag{5.42}$$

where H^{nil} is a nilpotent element of $\hat{T}_J$.

Proof. This follows directly from equations (5.37) and (5.38). □

Therefore, the Abresch-Fischer charts are really only interesting on the underlying manifold $\mathcal{A}(\mathcal{M})(\mathbb{R})$. For the higher points $\mathcal{A}(\mathcal{M})(\Lambda)$, they simply directly identify all the points above some point $p \in \mathcal{A}(\mathcal{M})(\mathbb{R})$ with the part of its tangent space which is proportional to odd generators of Λ.

Chapter 6

Integrability of almost complex structures

The manifold $\mathcal{A}(\mathcal{M})$ constructed in the previous chapter comprises all almost complex structures that the supermanifold $\mathcal{M}$ can be endowed with. Any superconformal structure requires, however, a complex structure, i.e., an *integrable* almost complex structure. The goal of this chapter is therefore to state the conditions for the integrability of an almost complex structure on a supermanifold. These conditions are direct super analogues of the ordinary ones, as was shown by Vaintrob [Vai85]. Although we suspect that it exists we do not try to construct an explicit supermanifold structure on the sets of integrable structures $\mathcal{C}(\mathcal{M})(\Lambda)$.

6.1 General case

As remarked in Section 5.2.1, we do not consider odd almost complex structures — all structures appearing here are even. Let J be an almost complex structure on the supermanifold $\mathcal{M}$. Then, as ordinary complex geometry, J splits the complexified tangent bundle $\mathcal{TM}^{\mathbb{C}}$ into two eigendistributions:

$$\mathcal{TM}^{\mathbb{C}} = \mathcal{TM}^{1,0} \oplus \mathcal{TM}^{0,1}, \tag{6.1}$$

where

$$\begin{aligned} \mathcal{TM}^{1,0} &:= \{X \in \mathcal{TM}^{\mathbb{C}} | JX = iX\}, & (6.2)\\ \mathcal{TM}^{0,1} &:= \{X \in \mathcal{TM}^{\mathbb{C}} | JX = -iX\}. & (6.3) \end{aligned}$$

In complete analogy to classical almost complex geometry, this splitting induces a bigrading on the complex $\Omega^\bullet$ of differential forms. A form $\omega \in \Omega^n$ is a super-antisymmetric map (cf. Section 2.3.1)

$$\omega : \mathcal{TM}^{\otimes n} \to \mathcal{O}_{\mathcal{M}}. \tag{6.4}$$

One extends forms to $\mathcal{TM}^{\mathbb{C}}$ by $\mathbb{C}$-linearity. This yields the complexified sheaf of differential forms, which, when applied to sections of $\mathcal{TM}^{\mathbb{C}}$, take values in $\mathcal{O}_{\mathcal{M}} \otimes \mathbb{C}$. The splitting (6.1) allows us to then decompose any $\omega \in \Omega^n \otimes \mathbb{C}$ as

$$\omega = \omega^{0,n} + \omega^{1,n-1} + \ldots + \omega^{n,0}, \tag{6.5}$$

where each $\omega^{p,q}$ is a super-antisymmetric map

$$\omega^{p,q} : (\mathcal{TM}^{1,0})^{\otimes p} \otimes (\mathcal{TM}^{0,1})^{\otimes q} \to \mathcal{O}_{\mathcal{M}} \otimes \mathbb{C}. \tag{6.6}$$

Thus, the space of complexified n-forms decomposes as

$$\Omega^n \otimes \mathbb{C} = \bigoplus_{p+q=n} \Omega^{p,q}. \tag{6.7}$$

If we denote by $\pi_{p,q}$ the projection onto the summand with indices p, q, then we can define the super versions of the Dolbeault operators as

$$\begin{aligned} \partial &:= \pi_{p+1,q} \circ d && (6.8) \\ \bar{\partial} &:= \pi_{p,q+1} \circ d, && (6.9) \end{aligned}$$

where the exterior differential d has to be understood as being extended by $\mathbb{C}$-linearity to complexified forms.

The Nijenhuis tensor associated with an even almost complex structure J is also defined by exactly the same formula as in classical geometry. It is a (2-1)-tensor field which is most conveniently defined by the way it acts on an arbitrary pair X, Y of vector fields:

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]. \tag{6.10}$$

Then we have the following result.

Theorem 6.1.1 (Vaintrob, [Vai85]). *Let J be an almost complex structure on a supermanifold $\mathcal{M}$. Then the following conditions are equivalent:*

1. $N_J \equiv 0$,
2. $\mathcal{TM}^{1,0}$ *is a sheaf of Lie subalgebras of* $\mathcal{TM}^{\mathbb{C}}$,
3. $\mathcal{TM}^{0,1}$ *is a sheaf of Lie subalgebras of* $\mathcal{TM}^{\mathbb{C}}$,
4. $d = \partial + \bar{\partial}$, $\partial^2 = 0$, $\bar{\partial}^2 = 0$,
5. *There exists a torsion-free connection on $\mathcal{M}$ with respect to which J is horizontal.*

In case these conditions hold, J is called integrable.

In particular, the second and third conditions prove to be of great value for our later considerations. Of course, the term "integrable" was chosen to denote an almost complex structure which can be induced by a complex structure, i.e., a complex atlas on $\mathcal{M}$. This is expressed by

Theorem 6.1.2 (Vaintrob, [Vai85])**.** *Let J be an integrable almost complex structure on a smooth supermanifold $\mathcal{M}$. Then there exists a complex structure on $\mathcal{M}$ such that J coincides with the almost complex structures locally induced by the chart maps, i.e., it acts as the multiplication operator by i on the tangent bundle of the complex supermanifold $\mathcal{M}$.*

With these beautiful results at hand, we can directly start to investigate the conditions for an almost complex structure on a $2|2$ smooth supermanifold to be integrable.

6.2 The $2|2$-dimensional case

To find the set or, even better, supermanifold of integrable almost complex structures on an arbitrary almost complex supermanifold $\mathcal{M}$ is in general a rather tough job. Is it at least as hard as finding them for the underlying manifold. But in the smooth $2|2$-dimensional case that we are interested in, things are much easier.

Prop. 4.2.1 tells us that every complex $1|1$-dimensional supermanifold can be interpreted as a pair (M, L) consisting of a Riemann surface and a holomorphic line bundle L, and vice versa. This provides us at once with a description of an integrable almost complex structure $J \in \mathcal{A}(\mathcal{M})(\mathbb{R})$. Every smooth supermanifold of dimension $2|2$ can be realized as a smooth surface with a smooth real vector bundle of rank 2 (by Batchelor's theorem [Bat79]). Denote this bundle as $\pi : E \to M$. An integrable almost complex structure on $\mathcal{M}$ must turn both the total space E as well as the base M into complex manifolds in a manner such that π becomes holomorphic.

It may seem at first that we need two complex structures to accomplish this, one on E and one on M, which have to be compatible. But the fact that we start with a supermanifold $\mathcal{M}$ gives us additional data: a splitting of the tangent sheaf $\mathcal{TM} = \mathcal{TM}_{\bar{0}} \oplus \mathcal{TM}_{\bar{1}}$ into an even and an odd part. Reducing the structure sheaf $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_M$ turns $\mathcal{TM}$ into a sheaf of locally free modules over $\mathcal{O}_M$ of rank $2|2$. This sheaf can be considered as the total space of the rank 4 smooth vector bundle E over M by changing the parity of its odd generators. This vector bundle inherits the splitting $E = E_{\bar{0}} \oplus E_{\bar{1}}$ from the tangent sheaf $\mathcal{TM}$. An almost complex structure on $\mathcal{M}$ is then integrable if and only if it is both integrable on the total space E and when it is restricted to the subbundle $E_{\bar{0}}$ (which is, of course, the tangent bundle of the underlying surface M). This is, clearly, more restrictive than the conditions in four ordinary real dimensions. The additional condition will show up below when we determine the integrable deformations of an integrable structure.

We note here another important fact: the smooth vector bundle E of rank 2 which determines $\mathcal{M}$ possesses a fixed degree d which completely classifies it up to isomorphism. This degree is, along with the topological invariants of the surface, an additional topological invariant of $\mathcal{M}$. Since we are interested only in orientable vector bundles (only they can be made complex), we see that the

set of isomorphism classes of smooth supermanifolds $\mathcal{M}$ whose underlying manifold is some fixed closed surface M and which possess integrable almost complex structures is isomorphic to $\mathbb{Z}$. After having endowed $\mathcal{M}$ with a complex atlas, the degree becomes the degree of the line bundle L. The degree does not classify L up to isomorphism anymore, rather, the set of these isomorphism classes is isomorphic to $\mathrm{Jac}(M)$, the Jacobian variety of M. This will be further discussed in Chapter 8.

6.2.1 The tangent space to $\mathcal{C}(\mathcal{M})(\Lambda)$

From now on, we assume that J is an integrable almost complex structure on $\mathcal{P}(\Lambda) \times \mathcal{M}$, where $\mathcal{M}$ is a smooth $2|2$-dimensional supermanifold $\mathcal{M}$. This means that J turns $\mathcal{P}(\Lambda) \times \mathcal{M}$ into a family of complex supermanifolds parametrized by $\mathcal{P}(\Lambda)$. Denote by z, θ the local complex coordinates induced by J, and by x, y, ξ_1, ξ_2 the corresponding real coordinates, i.e. $z = x + iy$ and $\theta = \xi_1 + i\xi_2$. The almost complex structure can, in real coordinates, always be written locally as

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{6.11}$$

where each of the entries is to be understood as a 2×2-matrix of smooth superfunctions. A smooth superfunction is, in this case, a local section of $\mathcal{O}_{\mathcal{M}} \otimes \Lambda$. Let

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.12}$$

be an arbitrary endomomorphism of the tangent bundle, i.e., a section of the form $\mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{E}nd(\mathcal{T}\mathcal{M})$. For H to be an element of $T_J\mathcal{A}(\mathcal{M})(\Lambda)$, it is necessary and sufficient that $HJ + JH = 0$, which implies that H can be put into the form

$$H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{6.13}$$

It will be more convenient to work in the complex picture. We have

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_{\xi_1} \\ \partial_y \\ \partial_{\xi_2} \end{pmatrix} = \begin{pmatrix} \partial_z \\ \partial_\theta \\ \partial_{\bar{z}} \\ \partial_{\bar{\theta}} \end{pmatrix} \tag{6.14}$$

and therefore, in the J-eigenbasis of the complexified tangent bundle, H takes the form

$$\frac{1}{2}\begin{pmatrix} \mathbb{1} & -i\mathbb{1} \\ \mathbb{1} & i\mathbb{1} \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ i\mathbb{1} & -i\mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & A - iB \\ A + iB & 0 \end{pmatrix}. \tag{6.15}$$

A tangent vector H of J at $\mathcal{A}(\mathcal{M})(\Lambda)$ is therefore locally described by just 4 local sections of $\mathcal{O}_{\mathcal{M}} \otimes \mathbb{C} \otimes \Lambda$, two even and two odd ones. We will denote these four

components of $A - iB$ as

$$A - iB = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{6.16}$$

Note that α and δ are even, while β and γ are odd.

Now consider an infinitesimal deformation of J along H, i.e. set $J' = J + itH$ for a small real parameter t. Then the eigenbasis of J' is given by

$$\begin{aligned} \partial_{z'} &= (\mathbb{1} + \frac{t}{2}H)\partial_z \\ \partial_{\theta'} &= (\mathbb{1} + \frac{t}{2}H)\partial_\theta. \end{aligned} \tag{6.17}$$

To identify the tangent space $T_J\mathcal{C}(\mathcal{M})(\Lambda)$, we must find out whether J' is again integrable. Vaintrob's Theorem (Thm. 6.1.1) will be the crucial tool here.

Proposition 6.2.1. *Let $J' = J + itH$ be an infinitesimal deformation of an integrable almost complex structure J on $\mathcal{M}$, and let H be locally given in the form (6.16). Then J' is again integrable if and only if the coefficient functions of H satisfy*

$$\frac{\partial\beta}{\partial\bar{z}} = \frac{\partial\alpha}{\partial\bar{\theta}}, \qquad \frac{\partial\delta}{\partial\bar{z}} = \frac{\partial\gamma}{\partial\bar{\theta}} \tag{6.18}$$

$$\frac{\partial\beta}{\partial\bar{\theta}} = 0, \qquad \frac{\partial\delta}{\partial\bar{\theta}} = 0. \tag{6.19}$$

Proof. By Thm. 6.1.1, it is sufficient to check whether the new eigenbasis (6.17) is again closed under Lie bracket. Consider first

$$[\partial_{z'}, \partial_{\theta'}] = \frac{t}{2}\left((\partial_z\overline{\beta})\partial_{\bar{z}} + (\partial_z\overline{\delta})\partial_{\bar{\theta}} - (\partial_\theta\overline{\alpha})\partial_{\bar{z}} - (\partial_\theta\overline{\gamma})\partial_{\bar{\theta}}\right). \tag{6.20}$$

This bracket must produce vector fields proportional to $\partial_{z'}, \partial_{\theta'}$. Clearly, the only possibility to achieve this is that bracket vanishes, so we have to require

$$\frac{\partial\beta}{\partial\bar{z}} = \frac{\partial\alpha}{\partial\bar{\theta}}, \qquad \frac{\partial\delta}{\partial\bar{z}} = \frac{\partial\gamma}{\partial\bar{\theta}}. \tag{6.21}$$

Likewise, we obtain

$$[\partial_{\theta'}, \partial_{\theta'}] = \frac{t}{2}\left((\partial_\theta\overline{\beta})\partial_{\bar{z}} + (\partial_\theta\overline{\delta})\partial_{\bar{\theta}}\right), \tag{6.22}$$

which yields the conditions

$$\frac{\partial\beta}{\partial\bar{\theta}} = 0, \qquad \frac{\partial\delta}{\partial\bar{\theta}} = 0. \tag{6.23}$$

□

The last condition (6.23) would be absent if we would study the problem of four ordinary real dimensions, because the commutator of an even vector field with itself always vanishes. Its presence shows that an integrable almost complex structure on a supermanifold is a more special structure than just a complex structure on the total space of a smooth vector bundle over a surface (cf. also the discussion in the previous section). Prop. 6.2.1 also makes it clear that there exist non-integrable almost complex structures on a smooth 2|2-dimensional supermanifold.

Corollary 6.2.2. *The almost complex structure on $\mathcal{A}(\mathcal{M})$ can be restricted to an almost complex structure on $\mathcal{C}(\mathcal{M})$.*

Proof. The almost complex structure on $\mathcal{A}(\mathcal{M})(\Lambda)$ was given on the tangent space $T_J\mathcal{A}(\mathcal{M})(\Lambda)$ by

$$\begin{aligned} \Phi : T_J\mathcal{A}(\mathcal{M})(\Lambda) &\rightarrow T_J\mathcal{A}(\mathcal{M})(\Lambda) \qquad (6.24)\\ H &\mapsto JH. \qquad (6.25)\end{aligned}$$

Locally, this corresponds just to replacing the entries $\alpha, \beta, \gamma, \delta$ of H by $i\alpha, i\beta, i\gamma, i\delta$, and the conjugated ones by $-i\overline{\alpha}$, etc. Obviously, these functions also satisfy the conditions (6.18) and (6.19). □

The conditions (6.18) and (6.19) are linear: they determine subspaces in each tangent space $T_J\mathcal{A}(\mathcal{M})(\Lambda)$. In order to show that $\mathcal{C}(\mathcal{M})$ is also a supermanifold, we need to first show that the tangent spaces determined in Prop. 6.2.1 form a $\overline{\mathbb{C}}$-submodule of the $\overline{\mathbb{C}}$-module $\hat{T}_J$ (compare with Prop. 5.2.6) which served as a model space for $\mathcal{A}(\mathcal{M})$.

Proposition 6.2.3. *The integrable deformations (those satisfying (6.18) and (6.19)) of an integrable almost complex structure J form a superrepresentable $\overline{\mathbb{C}}$-module $\hat{T}_J^{int}$.*

Proof. The integrable deformations form linear subspaces of the spaces $T_J\mathcal{A}(\mathcal{M})(\Lambda)$, and Cor. 6.2.2 tells us that these spaces are in fact all complex. What remains to be shown is that the inclusion of these spaces into the spaces $T_J\mathcal{A}(\mathcal{M})(\Lambda)$ is a functor morphism. This is equivalent to showing that for each morphism $\varphi : \Lambda \rightarrow \Lambda'$, the restriction of $\hat{T}_J(\varphi)$ to the integrable deformations $\hat{T}_J^{int}(\Lambda)$ yields only integrable deformations in $\hat{T}_J^{int}(\Lambda')$.

Let H be an integrable deformation in $T_J\mathcal{A}(\mathcal{M})(\Lambda)$, and let $\varphi : \Lambda \rightarrow \Lambda'$ be some given morphism. The action of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\varphi)$ on H is described by (5.4), and the action of $\hat{T}_J(\varphi)$ is then just the projection of $\hat{\Gamma}(\mathcal{E}nd(\mathcal{T}\mathcal{M}))(\varphi)(H)$ onto the subspace of endomorphisms anticommuting with J. Let $\tau_1, \dots, \tau_n$ denote the generators of Λ. The dependence of H on Λ is encoded, in its local expression, by the dependence of the entries $\alpha, \beta, \gamma, \delta$ on the τ_i. Each of these is a local section of $\mathcal{O}_\mathcal{M} \otimes \Lambda$ (where Λ denotes here the locally constant sheaf with stalk Λ). Thus each of the functions can be written as

$$\alpha = \sum_{I \subset \{1,\dots,n\}} \tau_I \alpha_I, \qquad (6.26)$$

where the sum runs over all increasingly ordered subsets, τ_I is the product of the appropriate τ_i's and α_I is a local section of $\mathcal{O}_{\mathcal{M}}$ of parity $|I| + p(\alpha)$ (cf. also Thm. 7.2.6 and the definitions preceding it). Morphisms $\Lambda \to \Lambda'$ affect only the generators τ_i, not the coefficient functions α_I which contain the dependence of α on $z, \bar{z}, \theta, \bar{\theta}$. Therefore, if H satisfies the integrablility conditions (6.18) and (6.19) this property will be preserved under $\varphi : \Lambda \to \Lambda'$. $\square$

6.2.2 A supermanifold structure for $\mathcal{C}(\mathcal{M})$

To obtain a complex supermanifold $\mathcal{C}(\mathcal{M})$, we would now have to show that we can find a chart around every $J \in \mathcal{C}(\mathcal{M})(\mathbb{R})$ which is isomorphic to a domain in $\hat{T}_J^{int}$. Unfortunately, it does *not* suffice to restrict the Abresch-Fischer charts constructed in Chapter 5 to the submodule $\hat{T}_J^{int}$. The images of integrable deformations are not necessarily integrable almost complex structures under this map. This is not a problem of supergeometry, but occurs as well for ordinary almost complex manifolds of dimension ≥ 4.

It is therefore very difficult to directly find a chart on $\mathcal{C}(\mathcal{M})$, and indeed we did not succeed in finding one. On the other hand, it seems to be intuitively clear that $\mathcal{C}(\mathcal{M})$ should be a (super)manifold. Looking at an ordinary almost complex manifold M, the set of integrable almost complex structures decomposes into orbits of the diffeomorphism group. So if the diffeomorphisms act freely, the set of integrable structures is a (possibly infinite) union of sets which are diffeomorphic to $\mathrm{Diff}(M)$. Formally this may be considered as a manifold. However, for moduli questions this "construction" seems to be rather useless, since it is the structure of the set of Diff-orbits that we are interested in, and that is precisely the structure that we are neglecting in this way. We will not try to address this problem for the $2|2$-dimensional case. Instead, in the next Chapter we will determine the deformations which are both integrable and transversal to the action of the diffeomorphism supergroup, and then directly construct a patch of super Teichmüller space.

Chapter 7

The supergroup $\widehat{\mathcal{SD}}(\mathcal{M})$

As in the case of an ordinary manifold, the superdiffeomorphisms of a supermanifold $\mathcal{M}$ form a highly nontrivial supermanifold. Its exhaustive investigation lies far beyond the scope of this work and the present knowledge of its author, and many problems are unsolved even for the ordinary case. In this Chapter, we will only outline the structure of the diffeomorphism supergroup and its action on various tensor fields on $\mathcal{M}$. The basic ideas underlying the constructions are once more due to V. Molotkov [Mol84]. To describe $\widehat{\mathcal{SD}}(\mathcal{M})$ as an actual supermanifold, one would have to use Fréchet supercharts. We will not do this here, but rather content ourselves with the construction of $\widehat{\mathcal{SD}}(\mathcal{M})$ as a group object in $\mathsf{Sets}^{\mathsf{Gr}}$. For the Fréchet approach, see [Mol84], [Leiar]. The main goal of this Chapter is to provide the prerequisites for the study of the existence of slices for the pullback action of the identity component $\widehat{\mathcal{SD}_0}(\mathcal{M})$ of the diffeomorphism supergroup on the integrable almost complex structures $\mathcal{C}(\mathcal{M})$ on a given smooth supersurface.

7.1 Inner Hom-functors for SMan

7.1.1 Generators for SMan

The first step towards the definition of the diffeomorphism supergroup is the construction of an inner Hom-functor for supermanifolds. According to the adjunction formula 2.20, such an inner Hom-functor $\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N})$ has to satisfy

$$\mathrm{Hom}(\mathcal{T},\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N})) \cong \mathrm{Hom}(\mathcal{T}\times\mathcal{M},\mathcal{N}) \qquad \forall\,\mathcal{T}\in\mathsf{SMan}. \tag{7.1}$$

Since $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ was denoted as $SC^\infty(\mathcal{M},\mathcal{N})$, we follow the notation of [Mol84] and denote $\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N})$ as $\widehat{SC}^\infty(\mathcal{M},\mathcal{N})$.

Theorem 7.1.1 (Molotkov [Mol84]). *Consider the functor*

$$\begin{array}{rcl} F:\mathsf{Sets}^{\mathsf{SMan}^\circ} & \to & \mathsf{Sets}^{\mathsf{SPoint}^\circ} \\ \Phi & \mapsto & \Phi\big|_{\mathsf{SPoint}}, \end{array} \tag{7.2}$$

which simply restricts each functor Φ to the full subcategory SPoint°. Then there exists an isomorphism between the composition of functors

$$\mathsf{SMan} \xrightarrow{H_*} \mathsf{Sets}^{\mathsf{SMan}^\circ} \xrightarrow{F} \mathsf{Sets}^{\mathsf{SPoint}^\circ} \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{Gr}}, \tag{7.3}$$

where H_* is the Yoneda embedding, and the forgetful functor

$$N : \mathsf{SMan} \longrightarrow \mathsf{Man}^{\mathsf{Gr}} \longrightarrow \mathsf{Sets}^{\mathsf{Gr}} . \tag{7.4}$$

We do not want to prove this here, since it would require some more technicalities that we will not need for other purposes and instead rely on Molotkov's results [Mol84]. This theorem is the categorical analogue of the statement that the superpoints are generators for the category SMan, extending Thm. 3.3.3 to the infinite-dimensional case. A direct consequence is the following statement, which says that superpoints completely describe a supermanifold by their morphisms into it, as is the case for ordinary manifolds and the point $\operatorname{Spec}\mathbb{K}$. The difference is just that there is a whole $\mathbb{Z}_{\geq 0}$-family of superpoints, and each of them yields one set of points. The full information about the supermanifold is then stored in this whole tower of points and their functoriality.

Corollary 7.1.2 (Molotkov [Mol84])**.** *For every supermanifold $\mathcal{M}$ and every $\Lambda \in \mathsf{Gr}$, there exists an isomorphism*

$$\mathcal{M}(\Lambda) \cong SC^\infty(\mathcal{P}(\Lambda), \mathcal{M}). \tag{7.5}$$

Proof. By applying the forgetful functor N (7.4), a supermanifold $\mathcal{M}$ can be seen as a functor $\mathsf{Sets}^{\mathsf{Gr}}$. The Yoneda embedding maps $\mathcal{M}$ to the functor $\mathrm{Hom}_{\mathsf{SMan}}(-, \mathcal{M})$ in $\mathsf{Sets}^{\mathsf{SMan}^\circ}$, which we can turn into a functor $\mathsf{Sets}^{\mathsf{SPoint}^\circ}$ by restriction (i.e., by applying the functor F from (7.2). Then Theorem 7.1.1 asserts that for every $\Lambda \in \mathsf{Gr}$, there exists an isomorphism of sets

$$SC^\infty(\mathcal{P}(\Lambda), \mathcal{M}) = \mathrm{Hom}_{\mathsf{SMan}}(\mathcal{P}(\Lambda), \mathcal{M}) \cong \mathcal{M}(\Lambda). \tag{7.6}$$

□

7.1.2 The inner Hom-functor

The inner Hom-functor for supermanifolds was already mentioned in an informal way in Sections 3.3.1 and 3.3.3 (see (3.39)). Thm. 7.1.1 now legitimates the following definition.

Definition 7.1.3. *Let $\mathcal{M}, \mathcal{N}$ be two supermanifolds. We define the inner Hom-functor $\widehat{SC}^\infty(\mathcal{M}, \mathcal{N})$ in $\mathsf{Sets}^{\mathsf{Gr}}$ by setting*

$$\widehat{SC}^\infty(\mathcal{M}, \mathcal{N})(\Lambda) := SC^\infty(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{N}) \tag{7.7}$$

for all $\Lambda \in \mathsf{Gr}$. To each $\varphi : \Lambda \to \Lambda'$, assign

$$\begin{aligned} \widehat{SC}^\infty(\mathcal{M}, \mathcal{N})(\varphi) : \widehat{SC}^\infty(\mathcal{M}, \mathcal{N})(\Lambda) &\to \widehat{SC}^\infty(\mathcal{M}, \mathcal{N})(\Lambda') \\ \sigma &\mapsto \sigma \circ (\mathcal{P}(\varphi) \times \mathrm{id}_{\mathcal{M}}). \end{aligned} \tag{7.8}$$

It is clear that this functor satisfies the adjunction formula (2.20), thus it is an inner Hom-functor. In general, it is not the inner Hom-functor in the category SMan, since it is usually impossible to give $\widehat{SC}^\infty(\mathcal{M},\mathcal{N})$ the structure of a Banach supermanifold. As for ordinary manifolds, the set of maps between them can usually only be endowed with the structure of a Fréchet manifold. We will not try to address these topological subtleties here, since in the case of the action of the superdiffeomorphism group $\widehat{\mathcal{SD}}(\mathcal{M})$ on $\mathcal{C}(\mathcal{M})$, they only play a role for the underlying manifold. Rather, we will deal with this case in a different way. For our purposes, it is sufficient to construct $\widehat{SC}^\infty(\mathcal{M},\mathcal{N})$ as an inner Hom-functor in $\mathsf{Sets}^{\mathsf{Gr}}$.

Note that the underlying set of $\widehat{SC}^\infty(\mathcal{M},\mathcal{N})$ is just

$$\widehat{SC}^\infty(\mathcal{M},\mathcal{N})(\mathbb{K}) = SC^\infty(\mathcal{P}(\mathbb{K})\times\mathcal{M},\mathcal{N}) \cong SC^\infty(\mathcal{M},\mathcal{N}), \tag{7.9}$$

since $\mathcal{P}(\mathbb{K})\times\mathcal{M}\cong\mathcal{M}$ by Lemma 3.6.8. Again, the underlying space of the inner-Hom functor consists precisely of the actual morphisms of the two objects, as was already the case for other super things, like super vector spaces.

7.1.3 Composition of morphisms

Let $\mathcal{M},\mathcal{M}',\mathcal{M}''$ be supermanifolds. Then there exists a composition map

$$\circ : \widehat{SC}^\infty(\mathcal{M},\mathcal{M}')\times\widehat{SC}^\infty(\mathcal{M}',\mathcal{M}'')\to\widehat{SC}^\infty(\mathcal{M},\mathcal{M}''), \tag{7.10}$$

which, of course, must be defined pointwise. Let $\Lambda\in\mathsf{Gr}$ be fixed, and let $f : \mathcal{P}(\Lambda)\times\mathcal{M}\to\mathcal{M}'$ and $g:\mathcal{P}(\Lambda)\times\mathcal{M}'\to\mathcal{M}''$ be two supersmooth maps. Then we define $g\circ f$ as the following composition:

$$g\circ f : \mathcal{P}(\Lambda)\times\mathcal{M}\xrightarrow{\mathrm{id}_{\mathcal{P}(\Lambda)}\times f}\mathcal{P}(\Lambda)\times\mathcal{M}'\xrightarrow{g}\mathcal{M}''\,. \tag{7.11}$$

On the functor $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})$ of supersmooth morphisms of $\mathcal{M}$ into itself, this composition is obviously associative. The following proposition also establishes the existence of a unit, making $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})$ into a monoid in $\mathsf{Sets}^{\mathsf{Gr}}$.

Proposition 7.1.4. *The functor morphism*

$$\begin{aligned} e:\mathcal{P}(\mathbb{R}) &\to \widehat{SC}^\infty(\mathcal{M},\mathcal{M}) && (7.12)\\ e_\Lambda:\{0\}=\mathcal{P}(\mathbb{R})(\Lambda) &\mapsto (\Pi_{\mathcal{M}}:\mathcal{P}(\Lambda)\times\mathcal{M}\to\mathcal{M}) && (7.13)\end{aligned}$$

in $\mathsf{Sets}^{\mathsf{Gr}}$ *is the unit for the composition* $\circ$ *defined in (7.11).*

Proof. This is clear from the definition (7.11). □

The unit element allows one to define inverse morphisms as follows.

Definition 7.1.5. *Let* $f\in\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\Lambda)$ *be a morphism* $f:\mathcal{P}(\Lambda)\times\mathcal{M}\to\mathcal{M}$. *The inverse* $f^{-1}:\mathcal{P}(\Lambda)\times\mathcal{M}\to\mathcal{M}$ *is defined to be the morphism such that*

$$(\mathrm{id}_{\mathcal{P}(\Lambda)}\times f)\circ f^{-1} = (\mathrm{id}_{\mathcal{P}(\Lambda)}\times f^{-1})\circ f = e_\Lambda(\{0\}) = \Pi_{\mathcal{M}}. \tag{7.14}$$

7.1.4 Geometric interpretation

An interpretation of the functor of points of a supermanifold was already sketched in Sections 3.3.1 and 3.3.3. It was argued that, for supermanifolds $\mathcal{M}, \mathcal{T}$, the $\mathcal{T}$-points $\mathrm{Hom}(\mathcal{T}, \mathcal{M})$ of $\mathcal{M}$ should be thought of as sections of the projection $\pi_{\mathcal{T}} : \mathcal{T} \times \mathcal{M} \to \mathcal{T}$. By Thm. 7.1.1 we know that only $\mathcal{P}(\Lambda)$-points need to be considered. If we assign to every $f \in \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M}')(\Lambda)$ a morphism

$$\Pi_{\mathcal{P}} \times f : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{P}(\Lambda) \times \mathcal{M}' \tag{7.15}$$

of families over $\mathcal{P}(\Lambda)$, then we clearly obtain a bijection

$$\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M}')(\Lambda) \to \mathrm{Hom}_{\mathsf{SMan}/\mathcal{P}(\Lambda)}(\mathcal{P}(\Lambda)^*(\mathcal{M}), \mathcal{P}(\Lambda)^*(\mathcal{M}')). \tag{7.16}$$

The functor $\mathcal{T}^* : \mathsf{SMan} \to \mathsf{SMan}/\mathcal{T}$ was defined in (3.26) as the map which assigns to every supermanifold $\mathcal{M}$ its trivial family over $\mathcal{T}$.

For $\mathcal{P}(\mathbb{R}) = \mathrm{Spec}\,\mathbb{R}$, everything reduces to the case of ordinary morphisms of supermanifolds, because $\mathcal{P}(\mathbb{R}) \times \mathcal{M} \cong \mathcal{M}$. The composition of morphisms consists in the composition of morphisms of families: let $f \in \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M}')(\Lambda)$ and $g \in \widehat{SC}^{\infty}(\mathcal{M}', \mathcal{M}'')(\Lambda)$ be given. Then their composition $g \circ f$ corresponds to

$$\begin{tikzcd} \mathcal{P}(\Lambda) \times \mathcal{M} \arrow[r, "\Pi_{\mathcal{P}(\Lambda)} \times f"] \arrow[dr, "\Pi_{\mathcal{P}(\Lambda)}"'] & \mathcal{P}(\Lambda) \times \mathcal{M}' \arrow[r, "\Pi_{\mathcal{P}(\Lambda)} \times g"] \arrow[d, "\Pi_{\mathcal{P}(\Lambda)}"] & \mathcal{P}(\Lambda) \times \mathcal{M}'' . \arrow[dl, "\Pi_{\mathcal{P}(\Lambda)}"] \\ & \mathcal{P}(\Lambda) & \end{tikzcd} \tag{7.17}$$

The fact that the composition $\circ$ is associative and has a unit can be seen as a consequence of this interpretation: the Λ-component of the functor morphism $\circ$ becomes identified with the ordinary composition of morphisms in the category $\mathsf{SMan}/\mathcal{P}(\Lambda)$, which is associative by definition. Inserting the unit of $\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M})(\Lambda)$ into the map (7.15), it becomes just the identity $\mathrm{id}_{\mathcal{P}(\Lambda)\times\mathcal{M}}$, as one expects for the neutral element of a space of morphisms.

7.2 The group of superdiffeomorphisms

From now on, we will only consider the case of a given finite-dimensional supermanifold $\mathcal{M}$.

Define for each $\Lambda \in \mathsf{Gr}$ a set $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ by setting

$$\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda) = \{f \in \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M})(\Lambda) \mid f \text{ invertible}\}. \tag{7.18}$$

Clearly, each of these sets is a group. Therefore if we can show that they form a functor in $\mathsf{Sets}^{\mathsf{Gr}}$, this functor will be a supergroup (a group object in $\mathsf{Sets}^{\mathsf{Gr}}$). To show that this is indeed the case and that the sets $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ actually form

the subfunctor in $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})$ whose underlying set is the group $\mathrm{Aut}(\mathcal{M})$, i.e., the isomorphisms of $\mathcal{M}$ in SMan, will be the goal of this section. Although a categorical proof exists [Mol84], we will study the elements of $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ directly as morphisms of ringed spaces, which will give us a somewhat deeper insight into their structure than a purely formal approach.

The first thing to show is that $\widehat{\mathcal{SD}}(\mathcal{M})$ is a subfunctor of $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})$. This means we have to prove that the restriction of the morphisms (7.8) to $\widehat{\mathcal{SD}}(\mathcal{M})$ is well-defined, i.e., that for each $\varphi:\Lambda\to\Lambda'$ the image of $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)|_{\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)}$ lies in $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda')$.

Proposition 7.2.1. *For each $\Lambda\in\mathsf{Gr}$ and each morphism $\varphi:\Lambda\to\Lambda'$, the restriction of $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)$ to $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ induces a group homomorphism*

$$\widehat{\mathcal{SD}}(\mathcal{M})(\varphi):\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)\to\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda'). \tag{7.19}$$

Proof. Applying the definition (7.8) to the neutral element $\Pi_{\mathcal{M}}:\mathcal{P}(\Lambda)\times\mathcal{M}\to\mathcal{M}$, we see immediately that

$$\Pi_{\mathcal{M}}\circ(\mathcal{P}(\varphi)\times\mathrm{id}_{\mathcal{M}})=\Pi_{\mathcal{M}}, \tag{7.20}$$

i.e., $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)$ maps the unit element to the unit element. Now let $f,g\in\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ be given. We have to show that

$$\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)(g\circ f)=(\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)(g))\circ(\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)(f)). \tag{7.21}$$

It is most insightful to compare the definition of the two functors. The left hand side corresponds to the composition

$$\mathcal{P}(\Lambda')\times\mathcal{M}\xrightarrow{(\mathcal{P}(\varphi),\mathrm{id}_{\mathcal{M}})}\mathcal{P}(\Lambda)\times\mathcal{M}\xrightarrow{(\Pi_{\mathcal{P}(\Lambda)},f)}\mathcal{P}(\Lambda)\times\mathcal{M}\xrightarrow{g}\mathcal{M}, \tag{7.22}$$

while the right hand side corresponds to

$$\begin{aligned}\mathcal{P}(\Lambda')\times\mathcal{M}&\xrightarrow{(\Pi_{\mathcal{P}(\Lambda')},\mathcal{P}(\varphi),\mathrm{id}_{\mathcal{M}})}\mathcal{P}(\Lambda')\times\mathcal{P}(\Lambda)\times\mathcal{M}\xrightarrow{(\Pi_{\mathcal{P}(\Lambda')},f)}\\ &\longrightarrow\mathcal{P}(\Lambda')\times\mathcal{M}\xrightarrow{(\mathcal{P}(\varphi),\mathrm{id}_{\mathcal{M}})}\mathcal{P}(\Lambda)\times\mathcal{M}\xrightarrow{g}\mathcal{M}\,.\end{aligned} \tag{7.23}$$

Let now $m\in\mathcal{M}(\Lambda'')$ be some Λ''-point of $\mathcal{M}$, $p\in\mathcal{P}(\Lambda')(\Lambda'')$ be a Λ''-point of $\mathcal{P}(\Lambda')$ and let $q\in\mathcal{P}(\Lambda)(\Lambda'')$ be its image under $\mathcal{P}(\varphi)$, i.e., $q=\mathcal{P}(\varphi)(p)$. Then (7.22) will map the pair (p,m) to

$$(p,m)\longmapsto(q,m)\longmapsto(q,f_{\Lambda''}(q,m))\longmapsto g(q,f_{\Lambda''}(q,m)). \tag{7.24}$$

On the other hand, (7.23) will map (p,m) as

$$(p,m)\mapsto(p,q,m)\mapsto(p,f_{\Lambda''}(q,m))\mapsto(q,f_{\Lambda''}(q,m))\mapsto g(q,f_{\Lambda''}(q,m))\,. \tag{7.25}$$

This shows that all components of the two functors (7.22) and (7.23) are indeed identical. □

Corollary 7.2.2. *$\widehat{\mathcal{SD}}(\mathcal{M})$ is a subfunctor of $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})$ and a group object in* $\mathsf{Sets}^{\mathsf{Gr}}$.

Proof. By Prop. 7.2.1, for $\varphi : \Lambda \to \Lambda'$, $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)$ maps invertible morphisms to invertible morphisms, so the restriction of $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)$ to $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ is well-defined. This means that the inclusion $\widehat{\mathcal{SD}}(\mathcal{M}) \subset \widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\varphi)$ is a functor morphism, and thus $\widehat{\mathcal{SD}}(\mathcal{M})$ is a subfunctor. Since each $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ is a group and each $\widehat{\mathcal{SD}}(\mathcal{M})(\varphi)$ is a group homomorphism, the second assertion is clear. □

7.2.1 Fine structure of supersmooth morphisms

We will now analyse the structure of a morphism $f : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ explicitly, i.e., by studying the supermanifolds involved as ringed spaces. This will give us a nice interpretation of the "higher points" $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\Lambda)$ in terms of odd parameters. As a byproduct, we find a factorization theorem for supersmooth morphisms.

In what follows we will keep Λ as

$$\Lambda = \Lambda_n = \mathbb{R}[\tau_1, \ldots, \tau_n].$$

Now, let $\varphi : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ be given. It consists of a continuous map

$$\phi : \{*\} \times M \to M \tag{7.26}$$

of the underlying topological spaces and a sheaf morphism

$$\varphi : \mathcal{O}_{\mathcal{M}} \to \phi_* \mathcal{O}_{\mathcal{P}(\Lambda_n)\times\mathcal{M}}, \tag{7.27}$$

which we also denote by φ. This should not cause any confusion, since the sheaf morphism is the only one we really have to deal with.[1] Then, for every topological point $p \in M$, φ consists of a stalk map

$$\varphi_p : \mathcal{O}_{\mathcal{M},\phi(p)} \to \mathcal{O}_{\mathcal{P}(\Lambda)\times\mathcal{M},(\{*\},p)}, \tag{7.28}$$

which is a homomorphism of superalgebras. Clearly,

$$\mathcal{O}_{\mathcal{P}(\Lambda)\times\mathcal{M},(\{*\},p)} \cong \Lambda \otimes_{\mathbb{R}} \mathcal{O}_{\mathcal{M},p} \tag{7.29}$$

since $\mathcal{P}(\Lambda)$ is just the one-point supermanifold with structure sheaf Λ. Let f be a germ in $\mathcal{O}_{\mathcal{M},\phi(p)}$. Then $\varphi_p(f)$ is of the general form

$$\varphi_p(f) = \sum_{I\subseteq\{1,\ldots,n\}} \tau_I \alpha_I(f). \tag{7.30}$$

[1] It can be shown [Leiar], [Var04] that the sheaf map in fact defines the continuous map of the underlying spaces, so one does not really have freedom in specifying it.

The sum runs over all increasingly ordered subsets, including the empty one. For a subset $I = \{i_1, \ldots, i_k\}$ with $i_1 < \ldots < i_k$, we write

$$\tau_I = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}. \tag{7.31}$$

The α_I are, for all I, homomorphisms of superalgebras

$$\alpha_I : \mathcal{O}_{\mathcal{M},\phi(p)} \to \mathcal{O}_{\mathcal{M},p}. \tag{7.32}$$

Each α_I has parity $|I|$, i.e., those whose index is of even length are even, while the others are odd. So, (7.30) reads explicitly as

$$\varphi_p(f) = \alpha_0(f) + \tau_1 \alpha_1(f) + \ldots + \tau_1 \cdots \tau_n \alpha_{1\ldots n}(f), \tag{7.33}$$

and the image has the same parity as f.

If we apply $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\epsilon_\Lambda)$ to φ, then we obtain a morphism

$$\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\epsilon_\Lambda)(\varphi) =: \varphi_{\mathbb{R}} : \mathcal{M} \to \mathcal{M},$$

i.e. a morphism of $\mathcal{M}$ into itself as a superringed space. We will call $\varphi_{\mathbb{R}}$ the *underlying morphism* of φ. It is of course not just a smooth selfmap of the underlying manifold M_{rd}, but a morphism of supermanifolds. Clearly, if φ has the form (7.30) on the stalk at p, then $\varphi_{\mathbb{R}}$ acts on this stalk simply as

$$\varphi_{\mathbb{R},p}(f) = \alpha_0(f), \tag{7.34}$$

since $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\epsilon_\Lambda)$ acts by annihilating all odd generators in the structure sheaf of $\mathcal{P}(\Lambda)$.

The space of all supersmooth morphisms $\mathcal{M} \to \mathcal{M}$ can have a quite complicated structure. Each set of higher points $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\Lambda)$, however, has a remarkably simple structure as a bundle over $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\mathbb{R})$. Since the proof of the result for general Λ is somewhat tedious and cumbersome, we will study the semigroups $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\Lambda_1)$ and $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\Lambda_2)$ separately first.

Proposition 7.2.3. *Let $\varphi : \mathcal{P}(\Lambda_1) \times \mathcal{M} \to \mathcal{M}$ be a Λ_1-point of $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})$. Then φ is uniquely determined by its underlying morphism $\varphi_{\mathbb{R}}$ and an odd vector field X on $\mathcal{M}$. Specifically,*

$$\varphi = \varphi_{\mathbb{R}} \circ (1 + \tau X), \tag{7.35}$$

where τ is the odd generator of Λ_1.

Proof. For any $p \in M$, we know by (7.30) that the induced stalk map $\varphi_{\phi(p)}$ acts on a germ of a function as

$$\varphi_{\phi(p)}(f) = \alpha_0(f) + \tau \alpha_1(f). \tag{7.36}$$

The homomorphism property for $\varphi_{\phi(p)}$ requires that for any two germs $f, g \in \mathcal{O}_{\mathcal{M},p}$, we must have

$$\varphi_{\phi(p)}(fg) = (\alpha_0(f) + \tau\alpha_1(f))(\alpha_0(g) + \tau\alpha_1(g)). \tag{7.37}$$

This requires

$$\begin{aligned} \alpha_0(fg) &= \alpha_0(f)\alpha_0(g), &(7.38)\\ \alpha_1(fg) &= \alpha_1(f)\alpha_0(g) + (-1)^{p(f)}\alpha_0(f)\alpha_1(g). &(7.39)\end{aligned}$$

Therefore, α_0 is a homomorphism $\mathcal{O}_{\mathcal{M},\phi(p)} \to \mathcal{O}_{\mathcal{M},p}$ and determines $\varphi_{\mathbb{R},\phi(p)}$, as already shown in (7.34). α_1 is an odd derivation of $\mathcal{O}_{\mathcal{M},\phi(p)}$ composed with α_0, i.e., there exists a germ $X_{\phi(p)}$ of a smooth odd vector field at $\phi(p)$ such that

$$\alpha_1(f) = \alpha_0(X_{\phi(p)}(f)). \tag{7.40}$$

Since all stalk maps have to be induced by a homomorphism of sheaves of smooth functions, the germs $X_{\phi(p)}$ must be induced by a global smooth odd vector field X on $\mathcal{M}$. □

Prop. 7.2.3 exhibits a factorization propery of morphisms of supermanifolds: each Λ_1-point of $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})$ is an morphism of $\mathcal{M}$ into itself composed with odd derivation of each stalk. This pattern will also show up in the general case.

Proposition 7.2.4. *Let $\varphi : \mathcal{P}(\Lambda_2) \times \mathcal{M} \to \mathcal{M}$ be a Λ_2-point of $\widehat{SC}^\infty(\mathcal{M})$. Then φ is uniquely determined by its underlying morphism $\varphi_{\mathbb{R}}$, two odd vector fields X_1, X_2 and an even vector field X_{12} on $\mathcal{M}$, such that*

$$\varphi = \varphi_{\mathbb{R}} \circ \exp(\tau_1 X_1 + \tau_2 X_2 + \tau_1\tau_2 X_{12}), \tag{7.41}$$

where τ_1, τ_2 are the odd generators of Λ_2.

Proof. By (7.30) we conclude again that the stalk map at $p \in M$ has the form

$$\varphi_{\phi(p)}(f) = \alpha_0(f) + \tau_1\alpha_1(f) + \tau_2\alpha_2(f) + \tau_1\tau_2\alpha_{12}(f). \tag{7.42}$$

Using the homomorphism property $\varphi_{\phi(p)}(fg) = \varphi_{\phi(p)}(f)\varphi_{\phi(p)}(g)$, we obtain an equation for each of the α_I. For α_0, we again obtain (7.38), and for α_1, α_2 an equation of the form (7.39). Thus again, α_0 can be identified with $\varphi_{\mathbb{R},\phi(p)}$, and α_1, α_2 correspond to odd vector fields X_1, X_2 composed with α_0. Finally, for α_{12}, we obtain

$$\begin{aligned}\alpha_{12}(fg) = \alpha_{12}(f)\alpha_0(g) + \alpha_0(f)\alpha_{12}(g) + (-1)^{p(\alpha_1(f))}\alpha_1(f)\alpha_2(g) -\\ - (-1)^{p(\alpha_2(f))}\alpha_2(f)\alpha_1(g). \quad (7.43)\end{aligned}$$

This is clearly satisfied for

$$\tau_1\tau_2\alpha_{12}(f) = \alpha_0(\tau_2 X_2(\tau_1 X_1(f))),$$

but also for

$$\tau_1\tau_2\alpha_{12}(f) = \alpha_0(\tau_1 X_1(\tau_2 X_2(f))).$$

Any linear combination

$$\tau_1\tau_2\alpha_{12} = c\cdot\alpha_0\circ(\tau_2 X_2(\tau_1 X_1)) + (1-c)\cdot\alpha_0\circ(\tau_1 X_1(\tau_2 X_2))$$

will therefore satisfy (7.43), as well. Moreover, from (7.43), we see that any two homomorphisms $\alpha_{12}, \alpha'_{12}$ which satisfy (7.43) are allowed to differ by an even derivation, since inserting them yields

$$(\alpha_{12}-\alpha'_{12})(fg) = (\alpha_{12}-\alpha'_{12})(f)\alpha_0(g) + \alpha_0(f)(\alpha_{12}-\alpha'_{12})(g)$$

as the necessary condition. Thus the most general form of $\tau_1\tau_2\alpha_{12}$ can be written as

$$\tau_1\tau_2\alpha_{12} = \frac{1}{2}\alpha_0\circ(\tau_1 X_1(\tau_2 X_2) + \tau_2 X_2(\tau_1 X_1)) + \tau_1\tau_2\alpha_0\circ X_{12}, \tag{7.44}$$

where X_{12} is the germ of a smooth even vector field at $\phi(p)$. Summing up (7.44) and $\tau_i\alpha_0\circ X_i$ for $i=1,2$ and α_0, we obtain

$$\alpha_0 + \tau_1\alpha_0\circ X_1 + \tau_2\alpha_0\circ X_2 + \frac{1}{2}(\tau_1\alpha_1(\tau_2\alpha_2) + \tau_2\alpha_2(\tau_1\alpha_1)) + \tau_1\tau_2\alpha_0\circ X_{12} \tag{7.45}$$

which matches exactly

$$\alpha_0\circ\exp(\tau_1 X_1 + \tau_2 X_2 + \tau_1\tau_2 X_{12}). \tag{7.46}$$

□

It is now quite clear what the general formula will look like, and which strategy has to be applied to prove it. Denote by $\mathfrak{S}(a_1\cdots a_n)$ the symmetrization of the ordered sequence $a_1\cdots a_n$, i.e.,

$$\mathfrak{S}(a_1\cdots a_n) = \frac{1}{n!}\sum_{\sigma\in P(n)} a_{\sigma(1)}\cdots a_{\sigma(n)},$$

where $P(n)$ is the group of permutations of n elements. To keep the notation simple, let us also introduce the following convention: the expression $I = I_1 + \ldots + I_j$ shall denote the decomposition of the ordered set I into an ordered j-tuple of subsets $I_1,\ldots,I_j$, each of which inherits an ordering from I. For example, $\{1,2\} = I_1 + I_2$ consists of the four partitions

$$\{\{\},\{1,2\}\},\quad \{\{1\},\{2\}\},\quad \{\{2\},\{1\}\},\quad \{\{1,2\},\{\}\}.$$

The notation $I = I_1\cup\ldots\cup I_j$, on the other hand, denotes the decomposition of the ordered set I into an *unordered* j-tuple of disjoint ordered subsets. So, $\{1,2\} = I_1\cup I_2$ consists of two partitions:

$$\{\{\},\{1,2\}\},\quad \{\{1\},\{2\}\}.$$

The following lemma will be useful.

Lemma 7.2.5. *Let A be an algebra, $f, g \in A$, and let $a_1, \ldots, a_n$ be derivations of A. Then*

$$\mathfrak{S}(a_1 \circ \ldots \circ a_n)(fg) = \sum_{\{1,\ldots,n\}=K+L} \mathfrak{S}(a_K)(f)\mathfrak{S}(a_L)(g), \tag{7.47}$$

where for $K = \{k_1, \ldots, k_j\}$, a_K denotes the composition

$$a_K = a_{k_1} \circ \ldots \circ a_{k_j}.$$

Proof. By the product rule, it is clear that $\mathfrak{S}(a_1 \circ \ldots \circ a_n)(fg)$ will take the form

$$\mathfrak{S}(a_1 \circ \ldots \circ a_n)(fg) = \sum_{\{1,\ldots,n\}=K+L} \frac{N(K,L)}{n!} a_K(f)a_L(g),$$

with some integer $N(K, L)$ denoting the multiplicity the K, L-summand. Since the symmetrized product on the left hand side contains all possible orderings of the operators a_i, all possible partitions of $\{1, \ldots, n\}$ into two ordered subsets will really appear on the right hand side. The summand with given K and L occurs exactly $(|K|+|L|)!/(|K|!|L|!)$ times, as one checks as follows: starting from an ordered sequence K of indices, there are $(|K| + |L|)!/|K|!$ ways to insert $|L|$ elements at arbitrary positions into it. But since the ordering of L is also fixed, one has to divide by the number of permutations of L. So we have

$$\begin{aligned}\mathfrak{S}(a_1 \circ \ldots \circ a_n)(fg) &= \sum_{K,L\subseteq\{1,\ldots,n\}} \frac{(|K| + |L|)!}{|K|!|L|!n!} a_K(f)a_L(g) &(7.48)\\ &= \sum_{\{1,\ldots,n\}=K+L} \mathfrak{S}(a_K)(f)\mathfrak{S}(a_L)(g) &(7.49)\end{aligned}$$

□

Theorem 7.2.6. *Let $\varphi : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ be a Λ-point of $\widehat{SC}^\infty(\mathcal{M}, \mathcal{M})$. Then φ is uniquely determined by its underlying morphism $\varphi_\mathbb{R} : \mathcal{M} \to \mathcal{M}$, as well as 2^{n-1} odd and $2^{n-1} - 1$ even vector fields X_I on $\mathcal{M}$ such that it can be expressed as*

$$\varphi = \varphi_\mathbb{R} \circ \exp\Big(\sum_{I\subseteq\{1,\ldots,n\}} \tau_I X_I\Big), \tag{7.50}$$

where the sum runs over all unordered nonempty subsets and τ_I is defined by (7.31).

Proof. For every $\phi(p) \in M$, we use (7.30) to write the stalk map $\varphi_{\phi(p)}$ as

$$\varphi_{\phi(p)} = \sum_{I\subseteq\{1,\ldots,n\}} \tau_I \alpha_I, \tag{7.51}$$

with τ_I defined by (7.31) and each α_I a homomorphism $\mathcal{O}_{\mathcal{M},\phi(p)} \to \mathcal{O}_{\mathcal{M},p}$. Since $\varphi_{\phi(p)}$ has to be a homomorphism, we require

$$\left(\sum_{K \subseteq \{1,\dots,n\}} \tau_K \alpha_K(fg) \right) = \left(\sum_{I \subseteq \{1,\dots,n\}} \tau_I \alpha_I(f) \right) \cdot \left(\sum_{J \subseteq \{1,\dots,n\}} \tau_J \alpha_J(g) \right). \tag{7.52}$$

Identifying (7.50) with the sum (7.51) rephrases the claim of the theorem as

$$\tau_I \alpha_I = \sum_{j=1}^{|I|} \sum_{I = I_1 \cup \ldots \cup I_j} \alpha_0 \circ \mathfrak{S}\left((\tau_{I_1} X_{I_1}) \circ \ldots \circ (\tau_{I_j} X_{I_j}) \right). \tag{7.53}$$

The summation runs over all partitions of I into unordered tuples of subsets, each subset inheriting an ordering from I (cf. the definition of the notation $I = I_1 \cup \ldots \cup I_j$ above). We will prove this formula by induction on $|I|$.

For indices I of length $|I| = 0, 1, 2$, the assertion holds by Props. 7.2.3 and 7.2.4. Assume the statement has been proven for indices up to length k. Then let $I = \{i_1, \dots, i_{k+1}\}$ be an index of length $k+1$. We must assure that (7.52) holds, which means we must find the general solution α_I for

$$\begin{aligned} \tau_I \alpha_I(fg) &= \alpha_0(f) \tau_I \alpha_I(g) + (-1)^{p(f)} \tau_I \alpha_I(f) \alpha_0(g) \\ & \sum_{\substack{I = K+L \\ K,L \neq \emptyset}} \tau_K \alpha_K(f) \tau_L \alpha_L(g). \end{aligned} \tag{7.54}$$

Since $|K|, |L| \leq k$, it follows that $\tau_K \alpha_K$ and $\tau_L \alpha_L$ must have the form (7.53). Therefore the sum in (7.54) can be written as

$$\begin{aligned} \sum_{\substack{I = K+L \\ K,L \neq \emptyset}} \alpha_0 \circ \Bigg(& \sum_{j=1}^{|K|} \sum_{K = K_1 \cup \ldots \cup K_j} \mathfrak{S}\left((\tau_{K_1} X_{K_1}) \circ \ldots \circ (\tau_{K_j} X_{K_j}) \right)(f) \circ \\ & \sum_{l=1}^{|L|} \sum_{L = L_1 \cup \ldots \cup L_l} \mathfrak{S}\left((\tau_{L_1} X_{L_1}) \circ \ldots \circ (\tau_{L_l} X_{L_l}) \right)(g) \Bigg). \end{aligned} \tag{7.55}$$

By Lemma 7.2.5, this equals

$$\sum_{j=2}^{|I|} \sum_{I = I_1 \cup \ldots \cup I_j} \alpha_0 \circ \mathfrak{S}\left((\tau_{I_1} X_{I_1}) \circ \ldots \circ (\tau_{I_j} X_{I_j}) \right)(fg). \tag{7.56}$$

The general solution to equation (7.54) therefore reads

$$\begin{aligned} \tau_I \alpha_I &= \alpha_0 \circ \tau_I X_I + \sum_{j=2}^{|I|} \sum_{I = I_1 \cup \ldots \cup I_j} \alpha_0 \circ \mathfrak{S}\left((\tau_{I_1} X_{I_1}) \circ \ldots \circ (\tau_{I_j} X_{I_j}) \right) \\ &= \sum_{j=1}^{|I|} \sum_{I = I_1 \cup \ldots \cup I_j} \alpha_0 \circ \mathfrak{S}\left((\tau_{I_1} X_{I_1}) \circ \ldots \circ (\tau_{I_j} X_{I_j}) \right), \end{aligned} \tag{7.57}$$

where X_I is a derivation of parity $|I|$ of $\mathcal{O}_{\mathcal{M},\phi(p)}$. □

7.2.2 The higher points of $\widehat{\mathcal{SD}}(\mathcal{M})$

As a corollary of the results of the previous section, we obtain the criterion for the invertibility of a supersmooth morphism $\mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$, and thus a statement on the structure of $\widehat{\mathcal{SD}}(\mathcal{M})$.

The underlying group $\widehat{\mathcal{SD}}(\mathcal{M})(\mathbb{R})$ of automorphisms of $\mathcal{M}$ as a smooth supermanifold can have a very complicated structure. Its topology is determined by the topology of $\mathrm{Diff}(M)$, i.e., of the diffeomorphism group of its underlying manifold, and by the topology of the spaces of isomorphisms of smooth vector bundles on M, since any smooth supermanifold can be realized as a smooth manifold and the exterior bundle of a smooth vector bundle on it (Batchelor's theorem [Bat79]). Even in the case of a $2|2$-dimensional smooth supersurface, the diffeomorphism group of M consists of infinitely many connected components generated from its identity component by applying the mapping class group of the surface. All these topological subtleties, however, pertain only to the underlying group $\widehat{\mathcal{SD}}(\mathcal{M})(\mathbb{R})$, while the higher points $\widehat{\mathcal{SD}}(\mathcal{M})(\Lambda)$ have a much simpler structure.

Theorem 7.2.7. *A supersmooth morphism $\varphi : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ is invertible if and only if its underlying morphism $\varphi_{\mathbb{R}} : \mathcal{M} \to \mathcal{M}$ is invertible. In this case, writing the sheaf map φ as*

$$\varphi = \varphi_{\mathbb{R}} \circ \exp\Big(\sum_{I\subseteq\{1,\ldots,n\}} \tau_I X_I\Big) \tag{7.58}$$

in accord with Thm. 7.2.6, its inverse is the sheaf map

$$\varphi^{-1} = \exp\Big(-\sum_{I\subseteq\{1,\ldots,n\}} \tau_I X_I\Big) \circ \varphi_{\mathbb{R}}^{-1}. \tag{7.59}$$

Proof. We have to show that for each stalk $\mathcal{O}_{\mathcal{M},p}$, the action of the composition of the two exponentials is the identity. Let φ be the germ of our morphism. We have to show that

$$\exp\Big(-\sum_{I\subseteq\{1,\ldots,n\}} \tau_I X_I\Big) \circ \exp\Big(\sum_{J\subseteq\{1,\ldots,n\}} \tau_J X_J\Big) = \mathrm{id}_{\mathcal{M},p}. \tag{7.60}$$

We can write

$$\exp\Big(-\sum_{I\subseteq\{1,\ldots,n\}} \tau_I X_I\Big) \circ \exp\Big(\sum_{J\subseteq\{1,\ldots,n\}} \tau_J X_J\Big) = 1 + \sum_K \tau_K \alpha_K \tag{7.61}$$

by expanding both exponentials. Using (7.53), we rewrite the expression on the

left hand side as

$$1+\left(\sum_{j=1}^{|I|}\sum_{I=I_1\cup\ldots\cup I_j}\mathfrak{S}\left((-\tau_{I_1}X_{I_1})\circ\ldots\circ(-\tau_{I_j}X_{I_j})\right)\right)\circ$$
$$\left(\sum_{k=1}^{|J|}\sum_{J=J_1\cup\ldots\cup J_k}\mathfrak{S}\left((\tau_{J_1}X_{J_1})\circ\ldots\circ(\tau_{J_k}X_{J_k})\right)\right) \quad (7.62)$$

Now $\tau_K\alpha_K$ on the right hand side of (7.61) is a sum over all partitions of K into ordered tuples of subsets. Pick one such tuple $\{K_1,\ldots,K_n\}$; the tuple, and each of the K_i, is ordered, and their union is K. On the left hand side, we have the corresponding sum

$$\frac{1}{k!(n-k)!}\sum_{k=0}^{n}(-1)^k(\tau_{K_1}X_{K_1})\circ\ldots\circ(\tau_{K_n}X_{K_n}) \quad (7.63)$$

of all ways of realizing this sequence of indices by contributions from either two of the exponentials in (7.60). But

$$\sum_{k=0}^{n}\frac{1}{k!(n-k)!}(-1)^k=\frac{1}{n!}(1+(-1))^n=0.$$

Therefore, each α_K on the right hand side of (7.61) receives only vanishing contributions, and thus (7.60) holds. □

Corollary 7.2.8. *The supergroup $\widehat{\mathcal{SD}}(\mathcal{M})$ is the restriction of $\widehat{SC}^\infty(\mathcal{M},\mathcal{M})$ onto* $\mathrm{Aut}(\mathcal{M})\subset\widehat{SC}^\infty(\mathcal{M},\mathcal{M})(\mathbb{R})$.

Proof. This is a direct consequence of Thm. 7.2.7. □

Moreover, we can now phrase the factorization properties of $\widehat{\mathcal{SD}}(\mathcal{M})$ as follows. Let $\mathcal{X}(\mathcal{M})$ denote the superrepresentable $\overline{\mathbb{R}}$-module of smooth sections of the tangent bundle of $\mathcal{M}$ (it exists due to Thm. 5.1.1). By the above discussion it is clear that we obtain a unipotent group $\mathcal{N}_{\mathcal{M}}$ by

$$\begin{aligned}\exp:\mathcal{X}(\mathcal{M})^{nil} &\to \mathcal{N}_{\mathcal{M}} &(7.64)\\ X &\mapsto \exp(X) &(7.65)\end{aligned}$$

because any $X\in\mathcal{X}(\mathcal{M})(\Lambda)$, $\Lambda\neq\mathbb{R}$ can be written as a sum

$$\sum_{I\subseteq\{1,\ldots,n\}}\tau_I X_I, \quad (7.66)$$

where again the τ_i are the free odd generators of Λ and each X_I is a vector field of parity $|I|$.

Theorem 7.2.9. *The supergroup $\widehat{\mathcal{SD}}(\mathcal{M})$ splits as a semidirect product*

$$\widehat{\mathcal{SD}}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M}) \ltimes \mathcal{N}_{\mathcal{M}}. \tag{7.67}$$

Proof. Obviously we have $\mathrm{Aut}(\mathcal{M}) \cap \mathcal{N}_{\mathcal{M}} = \{\mathrm{id}_{\mathcal{M}}\}$ and by Thm. 7.2.6 we know that $\widehat{\mathcal{SD}}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})\mathcal{N}_{\mathcal{M}}$. It remains to show that $\mathcal{N}_{\mathcal{M}}$ is normal. If $\varphi_{\mathbb{R}} \in \mathrm{Aut}(\mathcal{M})$ is given and X is a vector field, then we have for every germ f of a function on $\mathcal{M}$

$$X(\varphi^* f) = \varphi^* \circ D\varphi(X)(f). \tag{7.68}$$

If $\exp(\sum \tau_I X_I)$ is an element of $\mathcal{N}_{\mathcal{M}}$, this entails

$$\varphi_{\mathbb{R}} \circ \exp(\sum_{I \subseteq \{1,\ldots,n\}} \tau_I D\varphi_{\mathbb{R}}(X_I)) = \exp(\sum_{I \subseteq \{1,\ldots,n\}} \tau_I X_I) \circ \varphi_{\mathbb{R}}. \tag{7.69}$$

Since $D\varphi_{\mathbb{R}}$ is an isomorphism for an invertible $\varphi_{\mathbb{R}}$, this implies that

$$\varphi_{\mathbb{R}} \circ \mathcal{N}_{\mathcal{M}} = \mathcal{N}_{\mathcal{M}} \circ \varphi_{\mathbb{R}} \tag{7.70}$$

for all $\varphi_{\mathbb{R}} \in \mathrm{Aut}(\mathcal{M})$. □

7.2.3 The action of $\widehat{\mathcal{SD}}(\mathcal{M})$ on vector fields and 1-1-tensor fields

In order to analyse the structure of orbits of almost complex structures under pullback, we start by investigating the pushforward operation for vector fields, i.e., the differential of a superdiffeomorphism. The statement of Thm. 7.2.6 can be rephrased as follows: apart from the action of its underlying morphism, a diffeomorphism $\varphi : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ acts by Lie derivatives:

$$\varphi(f) = \varphi_{\mathbb{R}} \circ \exp(\sum_{I \subseteq \{1,\ldots,n\}} \tau_I L_{X_I})(f). \tag{7.71}$$

Intuitively, this corresponds to the fact that a deformation of a family of supermanifolds with fibre $\mathcal{M}$ along the odd dimensions of the base has to be linear with respect to the coordinate of this dimension. This in turn is due to the fact that one cannot "move a finite distance" along an odd direction: there is no way of localizing "at a point" in these directions. So all odd deformations have to be infinitesimal. We will, of course, discover the same phenomenon for the induced actions on vector fields and other tensors.

The Lie derivative of a vector field Y along a vector field X is defined as $L_X(Y) := [X, Y]$, where the bracket is the superbracket of vector fields. The Lie derivative of a (1-1)-tensor field σ along some even vector field X is then given by

$$\begin{aligned} (L_X\sigma)(Y) &:= -\sigma([X,Y]) + [X, \sigma(Y)] && (7.72) \\ &= [-\sigma \circ L_X + L_X \circ \sigma](Y), && (7.73) \end{aligned}$$

where Y is any vector field.

Proposition 7.2.10. *Let $\varphi : \mathcal{P}(\Lambda_n) \times \mathcal{M} \to \mathcal{M}$ be a Λ_n-point of $\widehat{\mathcal{SD}}(\mathcal{M})$ whose underlying diffeomorphism is $\varphi_{\mathbb{R}}$ and let Y a vector field on $\mathcal{P}(\Lambda) \times \mathcal{M}$. Suppose φ is given by an expression as in the statement of Thm. 7.2.6. Then we can write the differential $D\varphi$ as*

$$D\varphi(Y) = \exp(-\sum_{I\subseteq\{1,\ldots,n\}} \tau_I L_{X_I}) \circ D\varphi_{\mathbb{R}}(Y). \tag{7.74}$$

Proof. The differential is determined by the commutative diagram of sheaf maps

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{P}(\Lambda_n)\times\mathcal{M}} & \xrightarrow{\varphi} & \mathcal{O}_{\mathcal{P}(\Lambda_n)\times\mathcal{M}} \\ {\scriptstyle D\varphi(Y)}\downarrow & & \downarrow{\scriptstyle Y} \\ \mathcal{O}_{\mathcal{P}(\Lambda_n)\times\mathcal{M}} & \xrightarrow{\varphi} & \mathcal{O}_{\mathcal{P}(\Lambda_n)\times\mathcal{M}} \end{array}, \tag{7.75}$$

where we have extended φ to $\Pi_{\mathcal{P}(\Lambda)} \times \varphi$, i.e., to a morphism in $\mathsf{SMan}/\mathcal{P}(\Lambda)$. On the stalk at $\phi(p)$, we must then have

$$(D\varphi(Y))_{\phi(p)}(f) = (\varphi^{-1})_p \circ Y \circ (\varphi_{\phi(p)}(f)). \tag{7.76}$$

Inserting (7.59) and (7.50), this is equivalent to

$$D\varphi(Y) = \exp(-\sum_I \tau_I X_I) \circ \varphi_{\mathbb{R}}^{-1} \circ Y \circ \varphi_{\mathbb{R}} \circ \exp(\sum_I \tau_I X_I). \tag{7.77}$$

Clearly,

$$\varphi_{\mathbb{R}}^{-1} \circ Y \circ \varphi_{\mathbb{R}} = D\varphi_{\mathbb{R}}(Y). \tag{7.78}$$

Using the Baker-Campbell-Hausdorff formula and abbreviating

$$\mathcal{X} = \sum_I \tau_I X_I, \tag{7.79}$$

the expression (7.77) can be written as

$$D\varphi(Y) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \underbrace{[\mathcal{X}, [, \ldots, [\mathcal{X},}_{m\ times} D\varphi_{\mathbb{R}}(Y)] \cdots]]. \tag{7.80}$$

Expanding $\mathcal{X}$ into the sums (7.79) again, we obtain

$$D\varphi(Y) = \sum_{I\subseteq\{1,\ldots,n\}} \sum_{j=1}^{|I|} \sum_{I=I_1+\ldots+I_j} \frac{(-1)^j}{j!} [\tau_{I_1} X_{I_1}, \ldots, [\tau_{I_j} X_{I_j}, D\varphi_{\mathbb{R}}(Y)] \cdots]. \tag{7.81}$$

This is clearly the same as

$$\begin{aligned} D\varphi(Y) &= \sum_{I\subseteq\{1,\ldots,n\}} \sum_{j=1}^{|I|} \sum_{I=I_1\cup\ldots\cup I_j} \mathfrak{S}((-\tau_{I_1} L_{X_{I_1}}) \circ \ldots \circ (-\tau_{I_j} L_{X_{I_j}}))(D\varphi_{\mathbb{R}}(Y)) \\ &= \exp(-\sum_{I\subseteq\{1,\ldots,n\}} \tau_I L_{X_I}) \circ D\varphi_{\mathbb{R}}(Y). \end{aligned} \tag{7.82}$$

□

One might have guessed this from general arguments: quantities, which transform contravariantly, i.e. by pullback, should transform by application of a transformation $\exp(\sum_I \tau_I X_I)$. Those which transform covariantly, like vector fields, will transform inversely. We will need the analogous result for (1-1)-tensor fields.

Proposition 7.2.11. *Let $\varphi : \mathcal{P}(\Lambda) \times \mathcal{M} \to \mathcal{M}$ be a Λ-point of $\widehat{\mathcal{SD}}(\mathcal{M})$ whose underlying diffeomorphism is $\varphi_{\mathbb{R}}$, and let σ be an endomorphism of the tangent bundle, i.e. a 1-1-tensor field on $\mathcal{P}(\Lambda) \times \mathcal{M}$. Also, suppose that φ is given by an expression as in the statement of Thm. 7.2.6. Then we can express the pullback of $\varphi^*\sigma$ as*

$$\varphi^*\sigma = \varphi_{\mathbb{R}}^* \circ \exp(\sum_{I\subseteq\{1,\dots,n\}} \tau_I L_{X_I})(\sigma). \tag{7.83}$$

Proof. The pullback of a germ $\sigma \in \mathcal{E}nd(\mathcal{TM})_{(\{*\},\phi(p))}$ of an endomorphism of the tangent bundle of $\mathcal{P}(\Lambda_n) \times \mathcal{M}$ by a diffeomorphism φ is given by

$$(\varphi^*\sigma)_p = (D\varphi)^{-1}_{\phi(p)} \circ \sigma_{\phi(p)} \circ (D\varphi)_p. \tag{7.84}$$

Inserting (7.74) into this formula yields

$$(\varphi^*\sigma)_p = (D\varphi_{\mathbb{R}})^{-1} \circ \exp(\sum_{I\subseteq\{1,\dots,n\}} \tau_I L_{X_I}) \circ \sigma \circ \exp(-\sum_{J\subseteq\{1,\dots,n\}} \tau_J L_{X_J}) \circ (D\varphi_{\mathbb{R}}) \tag{7.85}$$

Therefore, the proposition is proven if we can show that

$$\exp(\sum_{I\subseteq\{1,\dots,n\}} \tau_I L_{X_I})(\sigma) = \exp(\sum_{I\subseteq\{1,\dots,n\}} \tau_I L_{X_I}) \circ \sigma \circ \exp(-\sum_{J\subseteq\{1,\dots,n\}} \tau_J L_{X_J}), \tag{7.86}$$

where on the left hand side, the Lie derivatives act on σ by (7.72), while on the right hand side, they have to be understood as acting on vector fields. Expanding the right hand side yields

$$\sum_{I\subseteq\{1,\dots,n\}} \sum_{j=1}^{|I|} \sum_{I=I_1+\dots+I_j} \sum_{a=0}^{j} \frac{(-1)^{j-a}}{a!(j-a)!} \cdot$$
$$\cdot L_{\tau_{I_1} X_{I_1}} \circ \dots \circ L_{\tau_{I_a} X_{I_a}} \circ \sigma \circ L_{\tau_{I_{a+1}} X_{I_{a+1}}} \circ \dots \circ L_{\tau_{I_j} X_{I_j}}, \tag{7.87}$$

while the left hand side can be written as

$$\sum_{I\subseteq\{1,\dots,n\}} \sum_{j=1}^{|I|} \sum_{I=I_1+\dots+I_j} \frac{1}{j!} (L_{\tau_{I_1} X_{I_1}}) \circ \dots \circ (L_{\tau_{I_j} X_{I_j}})(\sigma). \tag{7.88}$$

To see that (7.87) equals (7.88), we have to check which factor arises in front of the summand

$$L_{\tau_{I_1} X_{I_1}} \circ \dots \circ L_{\tau_{I_a} X_{I_a}} \circ \sigma \circ L_{\tau_{I_{a+1}} X_{I_{a+1}}} \circ \dots \circ L_{\tau_{I_j} X_{I_j}}$$

in (7.88) when one transforms it using the formula (7.72) for the Lie derivative of (1,1)-tensors. The factor contains $1/j!$ because the length of the summand is j, and a factor $(-1)^{j-a}$ as one can see from (7.72). But many summands of (7.88) contribute to it, namely each one that contains $I_1, \dots, I_a$ in this order and $I_{a+1}, \dots, I_j$ in the reverse order. There are precisely $j!/(j-a)!a!$ such summands, as one can see from the same reasoning as in the proof of Lemma 7.2.5. Thus, the factors of each summand in (7.87) and (7.88) coincide and (7.86) holds. □

7.3 The group $\mathrm{Aut}(\mathcal{M})$

The discussion of the previous sections was exclusively concerned with the identity element and the action of the "higher" points (i.e., those for $\Lambda \neq \mathbb{R}$) of the diffeomorphism supergroup. For the endeavour of quotienting out the action of $\widehat{\mathcal{SD}}_0(\mathcal{M})$, we also need to investigate the underlying ordinary group $\mathrm{Aut}(\mathcal{M})$. It turns out that its elements, again, can be split into a "hard" part, containing the analytic and topological subtleties typical of diffeomorphism groups, and a "nice" algebraic part. In this case, however, we cannot achieve a splitting as a semidirect product.

Let a finite-dimensional supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be given, let M be its underlying smooth manifold and let $E \to M$ be the associated smooth vector bundle, i.e., E is the locally free sheaf $\Pi(\mathcal{N}_{\mathcal{M}}/\mathcal{N}_{\mathcal{M}}^2)$ (cf. Section 2.2.6) with its action of $C^\infty(M) \cong C^\infty(\mathcal{M})/\mathcal{N}_{\mathcal{M}}$.

The tangent sheaf $\mathcal{TM}$ is, as a locally free module over $\mathcal{O}_{\mathcal{M}}$, filtered by the powers of the nilpotent ideal $\mathcal{N}_{\mathcal{M}}$. As in Section 2.2.6, we introduce the notation

$$\mathcal{TM}^{(j)}, \qquad j \in 2\mathbb{N}_0 \tag{7.89}$$

for the ideal in $\mathcal{TM}$ which consists of even vector fields which are of degree $\geq j$ in the odd coordinates of $\mathcal{M}$. That means that $X \in \mathcal{TM}^{(j)}$ can everywhere locally be written as

$$X = \sum_{k=j}^{n} \sum_{i,\alpha_1,\dots,\alpha_k} f_i^{\alpha_1\dots\alpha_k}(x)\theta_{\alpha_1}\cdots\theta_{\alpha_k}\frac{\partial}{\partial x_i} + \tag{7.90}$$

$$+\sum_{k=j}^{n-1} \sum_{n,\alpha_1,\dots,\alpha_{k+1}} f_n^{\alpha_1\dots\alpha_{k+1}}(x)\theta_{\alpha_1}\cdots\theta_{\alpha_{k+1}}\frac{\partial}{\partial\theta_n}, \tag{7.91}$$

with the $f_k^A(x)$ ordinary smooth functions of the even coordinates x_i on $\mathcal{M}$. The tangent space of $\widehat{\mathcal{SD}}(\mathcal{M})(\mathbb{R})$, i.e., the Lie algebra of $\mathrm{Aut}(\mathcal{M})$, is the algebra of even vector fields on $\mathcal{M}$. Since ordinary diffeomorphisms of the underlying manifold M are involved in any superdiffeomorphism, it is clear that the exponential map will not generate $\widehat{\mathcal{SD}}(\mathcal{M})(\mathbb{R})$. But this problem does not occur for nilpotent vector fields, which simplifies the situation considerably.

Lemma 7.3.1. *The Lie algebra $\mathcal{TM}^{(2)}$ generates a nilpotent group $N_{\mathcal{M}}$ of automorphisms of $\mathcal{M}$, which act as the identity on any associated vector bundle (M, E).*

Proof. Since the coefficient functions of any section X of $\mathcal{TM}^{(2)}$ are of degree ≥ 2 in the odd variable, the exponential $\exp(X)$ is a finite sum and therefore always well-defined. Additionally, one easily verifies that $\exp(X)\exp(-X) = \exp(-X)\exp(X) = \mathbb{1}$, because the vector field X is even and hence commutes with itself.

For any X, $\exp(X)$ acts on $\mathcal{O}_{\mathcal{M}}$ as an automorphism: its action (assuming it to be in the form (7.90)) can everywhere locally be expressed as

$$\begin{aligned} x_i &\rightarrow x_i + \sum_{j=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_{2j}} f_i^{\alpha_1\ldots\alpha_{2j}}(x)\theta_{\alpha_1}\cdots\theta_{\alpha_{2j}}, \\ \theta_j &\rightarrow \theta_j + \sum_{j=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_{2j+1}} f_n^{\alpha_1\ldots\alpha_{2j+1}}(x)\theta_{\alpha_1}\cdots\theta_{\alpha_{2j+1}}. \end{aligned} \tag{7.92}$$

Composing this map with the canonical epimorphism $q : \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}}/\mathcal{N}_{\mathcal{M}}^2$ evidently yields the identity. ☐

Therefore, the space of sections of the ideal subsheaf $\mathcal{TM}^{(2)} \subset \mathcal{TM}$ forms the Lie algebra associated with the nilpotent group $N_{\mathcal{M}}$, and generates this group by exponentiation. This leaves only the vector fields $\mathcal{TM}^{(0)}/\mathcal{TM}^{(2)}$. They form a quotient algebra, since the bracket of two vector fields of degree zero is again of degree zero. It is clear that they must be the infinitesimal transformations associated with morphisms of the associated vector bundles. To express this rigorously, we have to fix some notation.

Definition 7.3.2. *Let $p : E \rightarrow M$ be a smooth vector bundle. Let $\mathrm{Aut}_{\mathsf{VBun}}(M, E)$ denote the automorphism group of (M, E) as a vector bundle, i.e., each of its elements is a pair of maps $(f : E \rightarrow E, g : M \rightarrow M)$ satisfying*

1. *f, g are smooth and invertible,*
2. *they are compatible with the projection, i.e., $p \circ f = g \circ p$, and*
3. *f is fiberwise a linear automorphism of vector spaces $f_x : E_x \rightarrow E_{g(x)}$.*

The group $\mathrm{Aut}_{\mathsf{VBun}(M)}(E)$ is the group of automorphisms of E as a vector bundle over M, i.e., it consists of all elements of the form $(f, \mathrm{id}_M) \in \mathrm{Aut}_{\mathsf{VBun}}(M, E)$.

Lemma 7.3.3. *$\mathrm{Aut}_{\mathsf{VBun}(M)}(E)$ is a normal subgroup of $\mathrm{Aut}_{\mathsf{VBun}}(M, E)$, and the latter group splits as a semidirect product*

$$\mathrm{Aut}_{\mathsf{VBun}}(M, E) = \mathrm{Diff}(M) \ltimes \mathrm{Aut}_{\mathsf{VBun}(M)}(E). \tag{7.93}$$

Proof. Let $(f : E \to E, g : M \to M) \in \mathrm{Aut}_{\mathsf{VBun}}(M, E)$ be given. Then this statement is a direct consequence of the well-known fact that the map $f : E \to E$ can always be factorized as $f = g_* \circ h$ [Wel73], where $g_* : g^*E \to E$ is the canonical projection of the pullback, and $h : E \to E$ is a morphism of vector bundles with $p \circ h = p$. □

Then $\mathrm{Aut}(\mathcal{M})$ obviously "consists of" vector bundle morphisms and morphisms inducing higher nilpotent corrections. Here, however, we cannot obtain a splitting into a semidirect product.

Theorem 7.3.4. *Let $\mathcal{M}$ be a smooth supermanifold and let (M, E) be the associated canonical vector bundle over the underlying manifold M. Then the automorphism group $\mathrm{Aut}(\mathcal{M})$ fits into an exact sequence*

$$1 \longrightarrow N_{\mathcal{M}} \longrightarrow \mathrm{Aut}(\mathcal{M}) \longrightarrow \mathrm{Aut}_{\mathsf{VBun}}(M, E) \longrightarrow 1 \tag{7.94}$$

where $N_{\mathcal{M}}$ is the nilpotent group constructed in Lemma 7.3.1.

Proof. From our discussion above it should be clear that the elements of $N_{\mathcal{M}}$ are precisely those automorphisms of $\mathcal{M}$ which induce the identity on the associated vector bundle (M, E). Since there is no canonical map $\mathcal{M} \to (M, \wedge^\bullet E)$ we do not obtain a map $\mathrm{Aut}_{\mathsf{VBun}}(M, E) \to \mathrm{Aut}(\mathcal{M})$ which we would need in order to split $\mathrm{Aut}(\mathcal{M})$ into a semidirect product again. □

Chapter 8

Super Teichmüller spaces

In this Chapter we will investigate whether one can divide out the action of $\widehat{\mathcal{SD}}_0(\mathcal{M})$ on the integrable almost complex structures. All Riemann surfaces and supermanifolds whose base is a Riemann surface are assumed to be compact. Following Fischer and Tromba [Tro92], the goal would be to find local slices for the pullback action. These would constitute patches of the Teichmüller space $\mathcal{T}_{\mathfrak{vect}^L(1|1)}(\mathcal{M})$ of $\mathfrak{vect}^L(1|1)$-structures on a given smooth compact closed supersurface $\mathcal{M}$ of real dimension $2|2$, i.e., the Teichmüller space of supercomplex structures in $2|2$ real dimensions. It will turn out that this is not possible in the same way as in ordinary Teichmüller theory, since there remains a nontrivial subgroup of automorphisms in the identity component $\widehat{\mathcal{SD}}_0(\mathcal{M})$ of the diffeomorphism supergroup. The action of these automorphisms cannot be divided out from $\mathcal{C}(\mathcal{M})$ without destroying the supermanifold structure of the quotient. We will construct instead a supermanifold $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ which parametrizes the supercomplex structures up to these automorphisms. This can then be at best a semiuniversal family, but we will not try to prove semiuniversality here since it would lead us deep into deformation theory. The underlying manifold of $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$, however, can be shown to be universal among all families of compact complex supermanifolds of dimension $1|1$ *over a purely even base*. This is a simple corollary of the fact that every complex $1|1$-dimensional supermanifold is equivalent to a Riemann surface and a line bundle.

For the $\mathfrak{k}^L(1|1)$-structures, we can restrict this construction to spin curves. We construct a supermanifold $\mathcal{T}^{g}_{\mathfrak{k}^L(1|1)}$ whose base is the Teichmüller space of spin curves. Again, this supermanifold does not parametrize a universal family of $\mathfrak{k}^L(1|1)$-structures, since there remains a $\mathbb{Z}_2$-ambiguity stemming from the diffeomorphism $\theta \to -\theta$ of the supersurface, which is an automorphism of the $\mathfrak{k}^L(1|1)$-structure. $\mathcal{T}^{g}_{\mathfrak{k}^L(1|1)}$ is shown to be a closed subsupermanifold of $\mathcal{T}^{g,g-1}_{\mathfrak{vect}^L(1|1)}$. The remaining $\mathbb{Z}_2$-symmetry in the construction of the Teichmüller space of super Riemann surfaces was already observed by several authors, e.g., [CR88], [LR88]. In the latter reference, the authors remove the remaining automorphisms at the cost of destroying the supermanifold structure. They call the resulting space a

"canonical superorbifold". We do not pursue this idea here, but instead content ourselves with the construction of $\mathcal{T}^g_{\mathfrak{k}^L(1|1)}$ as the base of a semiuniversal family.

8.1 The space of nontrivial deformations

Throughout this and the next sections, we will study neighbourhoods in $\mathcal{A}(\mathcal{M})$ of a fixed integrable almost complex structure J on $\mathcal{M}$. In order to try to take a local quotient for the $\widehat{\mathcal{SD}}_0(\mathcal{M})$-action on a neighbourhood of J, we need to determine a direct sum decomposition of $\hat{T}_J^{int} = T_J\mathcal{C}(\mathcal{M}) = T_J\mathcal{S} \oplus \mathcal{L}J$, where $\mathcal{L}J$ comprises all integrable deformations of J which arise as Lie derivatives, i.e., deformations tangent to the $\widehat{\mathcal{SD}}_0(\mathcal{M})$-orbit. The quotient $T_J\mathcal{S}$ then represents the true deformations of J.

8.1.1 General setting

The problem is local. It is therefore most convenient to describe the tangent directions to J in the form (6.15) and (6.16), i.e., we write

$$H = \begin{pmatrix} 0 & A - iB \\ A + iB & 0 \end{pmatrix}, \tag{8.1}$$

where $A - iB$ is a square matrix of size $1|1$ (in the standard format) which we write in components as

$$A - iB = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{8.2}$$

The entries are local sections of $\mathcal{O}_{\mathcal{M}} \otimes \Lambda$ if $H \in \hat{T}_J^{int}(\Lambda)$. Of these four sections, α, δ are even and β, γ are odd. Let $\tau_1, \ldots, \tau_n$ be the free odd generators of $\Lambda = \Lambda_n$. This means that each component function, say α, can be written as an expansion

$$\alpha = \sum_{I \subset \{1,\ldots,n\}} \tau_I f_I, \tag{8.3}$$

where the sum runs over all increasingly ordered subsets and τ_I is the product of the corresponding τ_i's. Each f_I is a section of $\mathcal{O}_{\mathcal{M}}$ of parity $|I| + p(\alpha)$. On the other hand, since we have local complex coordinates z, θ, we can write each local smooth section f of $\mathcal{O}_{\mathcal{M}}$ as

$$f = \begin{cases} f_0(z) + f_3(z)\theta\bar{\theta} & \text{if } p(f) = \bar{0}, \\ f_1(z)\theta + f_2(z)\bar{\theta} & \text{if } p(f) = \bar{1}, \end{cases} \tag{8.4}$$

where the $f_i(z)$ are ordinary smooth functions on the underlying manifold M. The question to be answered is: for which tangent vectors H of J can we find an even real super vector field X on $\mathcal{M}$ such that

$$-iL_X J = H \tag{8.5}$$

holds, where L_X denotes the Lie derivative. Let us express X in the form

$$X = X^z \frac{\partial}{\partial z} + X^\theta \frac{\partial}{\partial \theta} + X^{\bar{z}} \frac{\partial}{\partial \bar{z}} + X^{\bar{\theta}} \frac{\partial}{\partial \bar{\theta}}, \tag{8.6}$$

where the coefficient functions are smooth real sections of $\mathcal{O}_\mathcal{M} \otimes \Lambda$.

Lemma 8.1.1. *Let X be an even smooth real vector field given in the form (8.6) and let $H \in \hat{T}_J^{int}(\Lambda)$ be a tangent vector to an integrable almost complex structure J. Then the condition $-iL_X J = H$ is equivalent to the following equations:*

$$2\frac{\partial X^z}{\partial \bar{z}} = \alpha, \qquad 2\frac{\partial X^\theta}{\partial \bar{z}} = \gamma \tag{8.7}$$

$$2\frac{\partial X^z}{\partial \bar{\theta}} = \beta, \qquad 2\frac{\partial X^\theta}{\partial \bar{\theta}} = \delta. \tag{8.8}$$

Proof. This is a straightfoward calculation in components. Write

$$J = i(\partial_z \otimes dz + \partial_\theta \otimes d\theta) - i(\partial_{\bar{z}} \otimes d\bar{z} + \partial_{\bar{\theta}} \otimes d\bar{\theta}). \tag{8.9}$$

The Lie derivative of a 1-1-tensor field on a super manifold is given by the same formula as for an ordinary manifold, except that a sign arises whenever two odd factors pass by each other. So, the $i\partial_z \otimes dz$-component of J contributes the terms

$$-\frac{\partial X^z}{\partial z}\partial_z \otimes dz - \frac{\partial X^\theta}{\partial z}\partial_\theta \otimes dz - \frac{\partial X^{\bar{z}}}{\partial z}\partial_{\bar{z}} \otimes dz - \frac{\partial X^{\bar{\theta}}}{\partial z}\partial_{\bar{\theta}} \otimes dz + \tag{8.10}$$

$$+\frac{\partial X^z}{\partial z}\partial_z \otimes dz + \frac{\partial X^z}{\partial \theta}\partial_z \otimes d\theta + \frac{\partial X^z}{\partial \bar{z}}\partial_z \otimes d\bar{z} + \frac{\partial X^z}{\partial \bar{\theta}}\partial_z \otimes d\bar{\theta}. \tag{8.11}$$

The terms proportional to $\partial_z \otimes dz$ cancel, and this will happen as well for the other contributions of the form $\partial_Z \otimes dZ$ for $Z = \bar{z}, \theta, \bar{\theta}$. Analogously, the term $-i\partial_{\bar{z}} \otimes d\bar{z}$ contributes

$$\frac{\partial X^z}{\partial \bar{z}}\partial_z \otimes d\bar{z} + \frac{\partial X^\theta}{\partial \bar{z}}\partial_\theta \otimes d\bar{z} - + \frac{\partial X^{\bar{z}}}{\partial \bar{z}}\partial_{\bar{z}} \otimes d\bar{z} + \frac{\partial X^{\bar{\theta}}}{\partial \bar{z}}\partial_{\bar{\theta}} \otimes d\bar{z} - \tag{8.12}$$

$$-\frac{\partial X^{\bar{z}}}{\partial z}\partial_{\bar{z}} \otimes dz - \frac{\partial X^{\bar{z}}}{\partial \theta}\partial_{\bar{z}} \otimes d\theta - - \frac{\partial X^{\bar{z}}}{\partial \bar{z}}\partial_{\bar{z}} \otimes d\bar{z} - \frac{\partial X^{\bar{z}}}{\partial \bar{\theta}}\partial_{\bar{z}} \otimes d\bar{\theta}. \tag{8.13}$$

The summands $i\partial_\theta \otimes d\theta$ and $-i\partial_{\bar{\theta}} \otimes d\bar{\theta}$ contribute similar expressions. Summing all up and equating this with H yields equations (8.8). $\square$

Just as one would expect, the vector field X is an infinitesimal automorphism of the complex structure (i.e., $L_X J = 0$) if and only if X^z and X^θ do not depend on $\bar{z}$ and $\bar{\theta}$, i.e., if they are superholomorphic.

8.1.2 The underlying tangent space

Let us start with the problem of determining the above mentionend decomposition for the underlying tangent space $\hat{T}_J^{int}(\mathbb{R}) = T_J\mathcal{S}(\mathbb{R}) \oplus \mathcal{L}J(\mathbb{R})$. In this case the analysis is simplified by the absence of any odd parameters τ_i. In this and the following sections we assume that the complex $1|1$-dimensional supermanifold $\mathcal{M}$ is given by a pair (M, L) where M is a closed, compact Riemann surface of genus $g \geq 2$ and L is a holomorphic line bundle of degree d on M (cf. Section 4.2.1).

Proposition 8.1.2. *The underlying tangent space $\hat{T}_J^{int}(\mathbb{R}) = T_J\mathcal{C}(\mathcal{M})(\mathbb{R})$ of an integrable almost complex structure J possesses a direct-sum decomposition*

$$\hat{T}_J^{int}(\mathbb{R}) = V \oplus \mathcal{L}J \tag{8.14}$$

where V is complex vector space of dimension $4g-3$, g is the genus of the underlying surface M and

$$\mathcal{L}J(\mathbb{R}) := \left\{ H \in \hat{T}_J^{int}(\mathbb{R}) \mid \exists \text{ a smooth even vector field } X \text{s.t. } L_X J = H \right\}. \tag{8.15}$$

Proof. Since no odd τ_i-parameters are involved, we can expand the coefficient functions $\alpha, \beta, \gamma, \delta$ of H as

$$\begin{aligned} \alpha &= \alpha_0(z) + \alpha_3(z)\theta\bar{\theta} && (8.16)\\ \beta &= \beta_1(z)\theta + \beta_2(z)\bar{\theta} && (8.17)\\ \gamma &= \gamma_1(z)\theta + \gamma_2(z)\bar{\theta} && (8.18)\\ \delta &= \delta_0(z) + \delta_3(z)\theta\bar{\theta} && (8.19) \end{aligned}$$

where each of the expansion coefficients is just a locally defined smooth function on the underlying manifold M. The integrability condition (6.19) implies $\beta_2 = 0 = \delta_3$. Likewise, we can expand each of the coefficient functions of a smooth even real vector field X as

$$\begin{aligned} X^z &= X_0^z(z) + X_3^z(z)\theta\bar{\theta} && (8.20)\\ X^\theta &= X_1^\theta(z)\theta + X_2^\theta(z)\bar{\theta} && (8.21) \end{aligned}$$

and similarly for $X^{\bar{z}}$ and $X^{\bar{\theta}}$.

Now we have to determine the obstructions to the solution of the equations (8.8). The equations

$$\beta = \beta_1\theta = 2\frac{\partial X^z}{\partial\bar{\theta}} = -X_3^z\theta \tag{8.22}$$

and

$$\delta = \delta_0 = 2\frac{\partial X^\theta}{\partial\bar{\theta}} = X_2^\theta \tag{8.23}$$

are purely algebraic, and thus can always be solved: for any given β_1, δ_0, we will find a vector field X that satisfies them.

So β and δ have been found, and condition (6.18) then entails that

$$\frac{\partial\alpha}{\partial\bar{\theta}} = -\alpha_3(z)\theta = \frac{\partial\beta}{\partial\bar{z}} = \frac{\partial\beta_1(z)}{\partial\bar{z}}\theta \tag{8.24}$$

$$\frac{\partial\gamma}{\partial\bar{\theta}} = \gamma_2(z) = \frac{\partial\delta}{\partial\bar{z}} = \frac{\partial\delta_0(z)}{\partial\bar{z}} \tag{8.25}$$

are also determined. The only equations that remain are

$$2\frac{\partial X_0^z}{\partial\bar{z}} \quad = \quad \alpha_0, \tag{8.26}$$

$$2\frac{\partial X_1^\theta}{\partial\bar{z}}\theta \quad = \quad \gamma_1\theta. \tag{8.27}$$

Let us first look at (8.26). The function $\alpha_0(z)$ is actually a smooth section of a holomorphic line bundle on M: it contributes the summand

$$\alpha = \alpha_0(z)\frac{\partial}{\partial z} \otimes d\bar{z}$$

to H. Therefore we can find a solution to (8.26) if and only if the Dolbeault cohomology class $[\alpha]$ vanishes. We can view α as a $\overline{\partial}$-closed smooth $(-1,1)$-form on M, and by the Hodge decomposition theorem, we know that any such space of forms can be directly decomposed as

$$\mathcal{A}^{p,q} = \overline{\partial}\mathcal{A}^{p,q-1} \oplus H \oplus \overline{\partial}^*\mathcal{A}^{p,q+1}. \tag{8.28}$$

Here, $\overline{\partial}^* = - * \overline{\partial} *$ is the adjoint of $\overline{\partial}$, $*$ is the Hodge star-operator (see, e.g., [Huy05],[GH94]) and H is the space of harmonic forms. Thus we can write

$$\alpha = \overline{\partial}\eta + \omega + \overline{\partial}^*\kappa, \tag{8.29}$$

but since $\overline{\partial}\alpha = 0$, we must have $\overline{\partial}^*\kappa = 0$. Therefore, the obstruction to the solution of (8.26) is

$$H_{\overline{\partial}}^{-1,1}(M) \cong H^1(M, K^{-1}) \cong H^0(M, K^2) \tag{8.30}$$

where the first "$\cong$" is due to the Dolbeault theorem, and second one follows from Serre duality. Moreover, this space splits off directly from $\hat{T}_J^{int}(\mathbb{R})$ as the subspace of those deformations for which α_0 is harmonic.

Now everything works completely analogously for equation (8.27). Here, however,

$$\gamma_1(z)\theta = (\gamma_1(z)\theta)\partial_\theta \otimes d\bar{z} \tag{8.31}$$

must be viewed as a smooth section of the line bundle $L \otimes L^{-1}$ since θ is a local section of L and ∂_θ is a section of $L^* \cong L^{-1}$. This means that the obstructions to the solution of (8.27) form the space

$$H^1(M, L \otimes L^{-1}) \cong H^1(M, \mathcal{O}) \cong H^0(M, K). \tag{8.32}$$

By the same argument as above, this space splits off directly from $\hat{T}_J^{int}$, and so altogether we can write

$$\hat{T}_J^{int}(\mathbb{R}) = H^0(M,K) \oplus H^0(M,K^2) \oplus \mathcal{L}J, \tag{8.33}$$

where the first two summands form a complex vector space of dimension $4g-3$. □

8.1.3 The Λ_1-points of $T_J\mathcal{S}$

Before treating the general Λ-points of $T_J\mathcal{S}$, it is instructive to first find the space $T_J\mathcal{S}(\Lambda_1)$ separately. The reason is that $T_J\mathcal{S}$ is supposed to be a super vector space, and therefore the sets of $\mathbb{R}$- and Λ_1-points contain the full information, while the higher points can be expected to be generated from these two spaces.

Proposition 8.1.3. *The Λ_1-tangent space $\hat{T}_J^{int}(\Lambda_1) = T_J\mathcal{C}(\mathcal{M})(\Lambda_1)$ of an integrable almost complex structure J possesses a direct-sum decomposition*

$$\hat{T}_J^{int}(\Lambda_1) = \overline{V}(\Lambda_1) \oplus \mathcal{L}J, \tag{8.34}$$

where V is a super vector space of dimension

$$\dim_{\mathbb{C}} V = (4g-3|4g-4+h^0(L^{-1})+h^0(K^{-1}\otimes L)) \tag{8.35}$$

and

$$\mathcal{L}J(\Lambda_1) := \left\{ H \in \hat{T}_J^{int}(\Lambda_1) \mid \exists \textit{ a smooth even vector field } X \textit{s.t. } L_X J = H \right\}. \tag{8.36}$$

Proof. Denote the odd generator of Λ_1 by τ. This time, each of the component functions of H can contain additional terms proportional to τ. So we have

$$\alpha = \alpha_0(z) + \tau\alpha_1(z)\theta + \tau\alpha_2(z)\bar{\theta} + \alpha_3(z)\theta\bar{\theta}, \tag{8.37}$$

$$\beta = \tau\beta_0(z) + \beta_1(z)\theta + \beta_2(z)\bar{\theta} + \tau\beta_3(z)\theta\bar{\theta}, \tag{8.38}$$

and analogously for γ and δ. For the same reason, the coefficients of a generic smooth even vector field X must be extended to

$$X^z = X_0^z(z) + \tau X_1^z(z)\theta + \tau X_2^z(z)\bar{\theta} + X_3^z(z)\theta\bar{\theta}, \tag{8.39}$$

$$X^\theta = \tau X_0^\theta(z) + X_1^\theta(z)\theta + X_2^\theta(z)\bar{\theta} + X_3^\theta(z)\theta\bar{\theta}, \tag{8.40}$$

and similarly for $X^{\bar{z}}$ and $X^{\bar{\theta}}$. As was already argued in Chapter 7, it may clarify things if one remembers that H, J and X are not really objects living on $\mathcal{M}$, but rather on the family $\mathcal{P}(\Lambda_1) \times \mathcal{M}$, i.e. they are deformations of vector fields and tensors on $\mathcal{M}$ along an odd dimension parametrized by τ.

Now the strategy closely follows that of Prop. 8.1.2. Note, first of all, that the integrability condition (6.19) tells us that

$$\beta_2 = \beta_3 = \delta_2 = \delta_3 = 0. \tag{8.41}$$

The equations

$$\beta = \tau\beta_0 + \beta_1\theta = 2\frac{\partial X^z}{\partial\bar\theta} = -\tau X_2^z - X_3^z\theta \tag{8.42}$$

and

$$\delta = \delta_0 + \tau\delta_1 = 2\frac{\partial X^\theta}{\partial\bar\theta} = X_2^\theta + \tau X_3^\theta\theta \tag{8.43}$$

are again purely algebraic and can always be solved: there always exists a vector field X satisfying them. The conditions (6.18) then fix α_2, α_3 and γ_2, γ_3. Thus the only remaining equations are

$$\begin{aligned} \alpha_0 + \tau\alpha_1\theta &= 2\frac{\partial X^z}{\partial\bar z} = 2\frac{\partial X_0^z}{\partial\bar z} + 2\tau\frac{\partial X_1^z}{\partial\bar z}\theta &(8.44)\\ \tau\gamma_0 + \gamma_1\theta &= 2\frac{\partial X^\theta}{\partial\bar z} = 2\tau\frac{\partial X_0^\theta}{\partial\bar z} + 2\frac{\partial X_1^\theta}{\partial\bar z}\theta. &(8.45)\end{aligned}$$

We can use arguments analogous to those of the proof of Prop. 8.1.2 to determine the obstructions to the solution of these equations. First of all, α_0 and γ_1 again contribute the space $H^0(M,K^2)\oplus H^0(M,K)$. But now there are also obstructions proportional to τ. In (8.44) this is $\alpha_1(z)\theta$, a smooth section of the line bundle $L\otimes K^{-1}$, therefore the odd obstructions to the solution of (8.44) are

$$H^1(M, L\otimes K^{-1}) \cong H^0(M, K^2\otimes L^{-1}), \tag{8.46}$$

where Serre duality has been used. Analogously, $\gamma_0(z)$ is a smooth section of L^{-1}, and therefore the odd obstructions to the solution of (8.45) are

$$H^1(M, L^{-1}) \cong H^0(M, K\otimes L). \tag{8.47}$$

By the Riemann-Roch theorem one finds

$$\begin{aligned} h^0(M,K\otimes L) - h^0(M,L^{-1}) &= \deg L + g - 1, &(8.48)\\ h^0(M,K^2\otimes L^{-1}) - h^0(M,K^{-1}\otimes L) &= 3g - 3 - \deg L. &(8.49)\end{aligned}$$

The Hodge decomposition theorem assures again that we can split off the obstructions as a direct submodule from $\hat T_J^{int}(\Lambda_1)$. Summarizing all the above, we have obtained a decomposition

$$\hat T_J^{int}(\Lambda_1) = H^0(M,K^2)\oplus H^0(M,K)\oplus\tau\left(H^0(M,K\otimes L)\oplus H^0(M,K^2\otimes L^{-1})\right)\oplus\mathcal{L}J, \tag{8.50}$$

and the first four summands clearly are the Λ_1-points of a super vector space V of the claimed dimension. □

8.1.4 The general case

It remains to be shown that we can indeed directly decompose the integrable deformations as $\hat T_J^{int} = T_J\mathcal{S}\oplus\mathcal{L}J$, where $T_J\mathcal{S}$ is a superrepresentable $\mathbb{C}$-module of finite dimension. Of course, $T_J\mathcal{S}$ will then have to coincide with $\overline{V}$, where V is the super vector space constructed in the proof of Prop. 8.1.3.

Theorem 8.1.4. *The $\overline{\mathbb{C}}$-module $\hat{T}_J^{int}$ tangent to J admits a direct sum decomposition*

$$\hat{T}_J^{int} = T_J\mathcal{S} \oplus \mathcal{L}J, \tag{8.51}$$

where $T_J\mathcal{S}$ is a superrepresentable $\overline{\mathbb{C}}$-module of dimension

$$(4g-3|4g-4+h^0(L^{-1})+h^0(K^{-1}\otimes L)),$$

and

$$\mathcal{L}J := \left\{ H \in \hat{T}_J^{int} \mid \exists \text{ a smooth even vector field } X s.t.\ L_X J = H \right\}. \tag{8.52}$$

Here, the definition of $\mathcal{L}J$ has to be understood pointwise: it contains all $H \in \hat{T}_J^{int}(\Lambda)$ for which there exists a smooth vector field on $\mathcal{P}(\Lambda)\times\mathcal{M}$ such that $L_X J = H$.

Proof. Assume that $\Lambda = \Lambda_n$ and denote the free odd generators of Λ by $\tau_1, \ldots, \tau_n$. In this setting, each of the component functions of H, for example α, can still be written as

$$\alpha = \alpha_0 + \alpha_1\theta + \alpha_2\bar{\theta} + \alpha_3\theta\bar{\theta}. \tag{8.53}$$

But now each of the expansion coefficients α_i can contain the odd parameters τ_i. Since α is even, we have

$$\alpha_0 = \alpha_{00} + \tau_i\tau_j\alpha_{0ij} + \ldots = \sum_{\substack{I\subset\{1,\ldots,n\}\\ |I| \text{ even}}} \tau_I\alpha_{0I} \tag{8.54}$$

The sum here runs over all increasingly ordered subsets of even cardinality, and τ_I is the product of the appropriate τ_i's in the same order. Each α_{0I} is just a local smooth function on the underlying manifold M. An analogous expansion exists for α_3. Since α_1, α_2 have to be odd, their expansions into powers of τ run over the subsets of odd cardinality.

Likewise, each of the coefficient functions $X^z, X^\theta, X^{\bar{z}}, X^{\bar{\theta}}$, has an expansion like (8.53) and each of the coefficients must then be expanded in a sum of the form (8.54), which runs over the subsets of even or odd cardinality, depending on the parity of the coefficient.

Again, the integrability conditions (6.19) tell us that $\beta_2 = \beta_3 = 0$ and $\delta_2 = \delta_3 = 0$. Also, the equations

$$\beta = \beta_0 + \beta_1\theta = 2\frac{\partial X^z}{\partial\bar{\theta}} = -X_2^z - X_3^z\theta \tag{8.55}$$

and

$$\delta = \delta_0 + \delta_1 = 2\frac{\partial X^\theta}{\partial\bar{\theta}} = X_2^\theta + X_3^\theta\theta \tag{8.56}$$

remain purely algebraic and can always be solved: there always exists a vector field X satisfying them. Note that we have suppressed the τ_i-expansions here and

in (8.57) and (8.58) below. The conditions (6.18) then still fix α_2, α_3 and γ_2, γ_3 completely. Thus, we are left with the same equations as before, namely

$$\alpha_0 + \alpha_1\theta = 2\frac{\partial X^z}{\partial \bar{z}} = 2\frac{\partial X_0^z}{\partial \bar{z}} + 2\frac{\partial X_1^z}{\partial \bar{z}}\theta \tag{8.57}$$

$$\gamma_0 + \gamma_1\theta = 2\frac{\partial X^\theta}{\partial \bar{z}} = 2\frac{\partial X_0^\theta}{\partial \bar{z}} + 2\frac{\partial X_1^\theta}{\partial \bar{z}}\theta. \tag{8.58}$$

But now each of the coefficients of the expansions (8.54) may contribute obstructions. The equation

$$\alpha_0 = 2\frac{\partial X_0^z}{\partial \bar{z}} \tag{8.59}$$

now actually expands into a system of 2^{n-1} equations, one for each increasingly ordered subset $I \subset \{1, \ldots, n\}$ of even cardinality. Each such equation has the form

$$\alpha_{0I} = 2\tau_I \frac{\partial X_{0I}^z}{\partial \bar{z}} \tag{8.60}$$

and thus each of these component equations contributes the space $H^0(M, K^2)$ as obstructions. Thus, the full space of obstructions for the α_0-equation is

$$\Lambda_{\bar{0}} \otimes H^0(M, K^2). \tag{8.61}$$

By the same arguments, the obstructions to the solution of the γ_1-equation are

$$\Lambda_{\bar{0}} \otimes H^0(M, K), \tag{8.62}$$

and those to the solution of the α_1-equation are

$$\Lambda_{\bar{1}} \otimes H^0(M, K^2 \otimes L^{-1}). \tag{8.63}$$

Finally, the obstructions for γ_0 are, not surprisingly,

$$\Lambda_{\bar{1}} \otimes H^0(M, K \otimes L). \tag{8.64}$$

The Hodge decomposition theorem again assures that we can split off the obstructions, component by component, as direct summands. Defining a complex super vector space

$$V = H^0(M, K^2) \oplus H^0(M, K) \oplus \Pi\left(H^0(M, K^2 \otimes L^{-1}) \oplus H^0(M, K \otimes L)\right), \tag{8.65}$$

this means that we have decomposed $\hat{T}_J^{int}(\Lambda)$ as

$$\hat{T}_J^{int}(\Lambda) = \overline{V}(\Lambda) \oplus \mathcal{L}J(\Lambda), \tag{8.66}$$

as was to be shown. □

8.1.5 Infinitesimal superconformal automorphisms

Before one can take the quotient $\mathcal{C}(\mathcal{M})/\widehat{\mathcal{SD}}_0(\mathcal{M})$, one has to check whether there are automorphisms of the supercomplex structure which are homotopic to the identity, i.e., whether there are global vector fields generating superconformal transformations. In the case of the $\mathfrak{vect}^L(1|1)$-structure, these would be just the global holomorphic sections of $\mathcal{TM}$. If such vector fields exist, they will form the Lie algebra of a subgroup $\mathrm{Aut}_0(J) \subset \widehat{\mathcal{SD}}_0(\mathcal{M})$ of superconformal automorphisms of J, which is the stabilizer subgroup of $J \in \mathcal{C}(\mathcal{M})$. Along the $\widehat{\mathcal{SD}}_0(\mathcal{M})$-orbit of J, this group changes, but its conjugacy class is preserved.

We directly deduce the following corollary from Lemma 8.1.1:

Corollary 8.1.5. *Let $\mathcal{M}$ be a complex $1|1$-dimensional supermanifold, and let (M, L) be the Riemann surface and the line bundle which are equivalent to $\mathcal{M}$ by Prop. 4.2.1. Then there exists a super vector space of infinitesimal automorphisms of the supercomplex structure of complex dimension*

$$h^0(\mathcal{TM}) = \dim_{\mathbb{C}} \mathrm{aut}_{\mathfrak{vect}^L(1|1)} = (1|h^0(L \otimes K^{-1}) + h^0(L^{-1})). \tag{8.67}$$

Proof. By Lemma 8.1.1, a vector field X has the property $L_X J = 0$ if and only if

$$2\frac{\partial X^z}{\partial \bar{z}} = 0, \qquad 2\frac{\partial X^\theta}{\partial \bar{z}} = 0$$
$$2\frac{\partial X^z}{\partial \bar{\theta}} = 0, \qquad 2\frac{\partial X^\theta}{\partial \bar{\theta}} = 0.$$

This means that X^z and X^θ have to be superholomorphic. The coefficient X_0^z is a section of K^{-1}, the sheaf of holomorphic tangent vector fields on M. If the genus of M is greater than one, this sheaf has no global holomorphic sections. The other even contribution to the infinitesimal automorphisms stems from the coefficient function $X_1^\theta \theta$, which is a section of $L \otimes L^{-1}$. Indeed, this bundle has holomorphic sections:

$$H^0(M, L \otimes L^{-1}) \cong H^0(M, \mathcal{O}_M) = \mathbb{C}. \tag{8.68}$$

This is the 1-dimensional even subspace of $\mathrm{aut}_{\mathfrak{vect}^L(1|1)}$. The odd infinitesimal automorphisms are afforded by $X_1^z \theta$, which contributes $H^0(M, L \otimes K^{-1})$, and by X_0^θ, which contributes $H^0(M, L^{-1})$. $\square$

For the odd vector fields, one notes that the two spaces of holomorphic sections whose sum makes up the odd infinitesimal automorphisms cannot be both nontrivial: if $h^0(L^{-1}) \geq 0$ then we must have $\deg(L) \leq 0$. But then $L \otimes K^{-1}$ cannot have any nontrivial holomorphic sections. There is, in fact, a range of degrees where no infinitesimal odd automorphisms exist at all, namely for

$$0 < \deg(L) < 2g - 2. \tag{8.69}$$

Interestingly, if L is a spin bundle, as is the case for a $\mathfrak{k}^L(1|1)$-structure, we are precisely in this range.

Geometrically, the odd infinitesimal automorphisms act as translations. If $h(z)$ is, e.g., a holomorphic section of L^{-1}, then we can generate a $0|1$-parameter subgroup from it which will locally act as $\exp(\tau h(z)\frac{\partial}{\partial\theta})$ (here, τ is the odd generator of Λ_1). The effect of this subgroup is a translation $\theta \mapsto \theta + \tau h(z)$. Likewise, given a holomorphic section $g(z)$ of $L \otimes K^{-1}$, we can generate a $0|1$-parameter subgroup which translates $z \mapsto z + \tau g(z)\theta$.

There is another reason to restrict the degree of L to the range (8.69). We have seen that the super Teichmüller space of $\mathfrak{vect}^L(1|1)$-structures (i.e., of supercomplex structures) has, if it exists, the odd dimension $h^0(M, K^2 \otimes L^{-1}) + h^0(M, K \otimes L)$. But for line bundles of certain degrees, these values *depend on the complex structure on* M [ACGH85]. A rough idea of the problem is provided by the Riemann-Roch theorem, which states that

$$h^0(M, L) - h^0(M, K \otimes L^{-1}) = \deg(L) - g + 1 \tag{8.70}$$

holds for a Riemann surface M of genus g and a holomorphic line bundle $L \to M$. Now, if we are interested in $h^0(M, L)$, then this formula is immediately useful only for $\deg(L) < 0$ (when there are no nontrivial sections), and for $\deg(L) > 2g - 2$, because then $h^0(M, K \otimes L^{-1}) = 0$. One might call this range of degrees the "topological range": $h^0(M, L)$ is then exclusively determined by the degree of L and the genus. But for degrees in the range (8.69), there is no way to simply read off $h^0(M, L)$ from the Riemann-Roch theorem, and indeed, the dimension in this case may depend on the complex structure of the surface. The dimension can jump when one varies within the moduli space. It is known that this can happen, for example, for spin bundles [ACGH85].

We must exclude this phenomenon for the bundles $K^2 \otimes L^{-1}$ and $K \otimes L$ in order to have a chance to define super Teichmüller space $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ at least as a quotient of a supermanifold. Therefore, from now on we restrict the degree of the line bundle L to which $\mathcal{O}_{\mathcal{M},\bar{1}}$ is equivalent to the range (8.69). This also rules out the occurrence of odd infinitesimal automorphisms. In this range, the tangent space of the slice always has the dimension $(4g - 3|4g - 4)$. In fact, one can extend the construction to bundles L of degree 0 if one excludes the case of trivial bundles, i.e., those isomorphic to $\mathcal{O}_M$ [Man85]. This corresponds to excluding the zero section of the family of Jacobians $J(V_g)$ (see below). While this case could be handled in theory, a thorough discussion is outside the scope of this work.

The even automorphisms cannot be disposed of so easily. They act by constant overall rescalings of the fibers of L. If $M \mapsto c \in \mathbb{C}$ is such a section, then we can generate a $1|0$-parameter family of automorphisms of the supercomplex structure whose action is locally given by

$$\exp(c\theta\frac{\partial}{\partial\theta}) = \sum_{n=0}^{\infty} \frac{c^n}{n!} \left(\theta\frac{\partial}{\partial\theta}\right)^n = \mathrm{e}^c\theta\frac{\partial}{\partial\theta}, \tag{8.71}$$

which simply maps $\theta \mapsto \mathrm{e}^c\theta$. This means that the vector fields $c\theta\frac{\partial}{\partial\theta}$ generate a subgroup of $\widehat{\mathcal{SD}}_0(\mathcal{M})$ which is isomorphic to $\mathbb{C}^\times$. The presence of these automor-

phisms is indeed a severe problem for the construction of the super Teichmüller space of supercomplex structures. In fact, the space $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ that we will construct below is *not* the base of a universal family. The reason is that $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ will still carry a $\mathbb{C}^\times$-action, and the elements of an orbit of this action all parametrize the same complex structure. The problem is passed along to the Teichmüller space $\mathcal{T}^{g}_{\mathfrak{k}^L(1|1)} \subset \mathcal{T}^{g,g-1}_{\mathfrak{vect}^L(1|1)}$, which will carry a residual $\mathbb{Z}_2$ action. To see the effect of the $\mathbb{C}^\times$-ambiguity, it is best to look directly at the cocycle defining the complex structure. Let $\{U_\alpha\}_{\alpha\in A}$ be a cover of the underlying topological space M of a complex supersurface. Then the cocycle defining the complex structure on $\mathcal{P}(\Lambda) \times \mathcal{M}$ is given by transition functions

$$\begin{aligned} z_\beta &= f_{\alpha\beta}(z_\alpha) + \psi_{\alpha\beta}(z_\alpha)\theta_\alpha \\ \theta_\beta &= \eta_{\alpha\beta}(z_\alpha) + g_{\alpha\beta}(z_\alpha)\theta_\alpha. \end{aligned} \tag{8.72}$$

Here f and g are even sections of $\mathcal{O}_\mathcal{M} \otimes \Lambda$ on $U_\alpha \cap U_\beta$, while ψ and η are odd sections. An automorphism $\theta \mapsto \mathrm{e}^c\theta$ rescales both θ_α and θ_β by e^c, therefore it alters the cocycle to

$$\begin{aligned} z_\beta &= f_{\alpha\beta}(z_\alpha) + \mathrm{e}^c\psi_{\alpha\beta}(z_\alpha)\theta_\alpha \\ \theta_\beta &= \mathrm{e}^{-c}\eta_{\alpha\beta}(z_\alpha) + g_{\alpha\beta}(z_\alpha)\theta_\alpha. \end{aligned} \tag{8.73}$$

The cocycles (8.72) and (8.73) describe the same supercomplex structure. However, the $\mathbb{C}^\times$-action does not affect the underlying cocycle consisting of f and g alone, which describes the complex structure of $\mathcal{M}$. Therefore, the underlying Teichmüller space is not affected by the automorphisms $\theta \mapsto \mathrm{e}^c\theta$, and will turn out to be a complex manifold. For the odd parameters of a deformation, we have found the following.

Theorem 8.1.6. *Let $\mathcal{M}$ be a complex $1|1$-dimensional supermanifold, and let (M, L) be the Riemann surface and the holomorphic line bundle to which $\mathcal{M}$ is equivalent. Define a $\mathbb{C}^\times$-action on the odd deformations of the complex structure J of $\mathcal{M}$ by*

$$\begin{aligned} \mathbb{C}^\times \times (H^0(M, K^2 \otimes L^{-1}) \oplus H^0(M, K \otimes L)) &\to H^0(M, K^2 \otimes L^{-1}) \oplus H^0(M, K \otimes L) \\ (\alpha, v, w) &\mapsto (\alpha v, \alpha^{-1} w). \end{aligned} \tag{8.74}$$

Then all points of an orbit of this $\mathbb{C}^\times$-action parametrize the same deformation of J.

Proof. Two deformations are equivalent if the complex structures J', J'' produced by them are the same, i.e., if they are related by an automorphism. A complex structure J on $\mathcal{M}$ is given by a cocycle of the form

$$\begin{aligned} z_\beta &= f(z_\alpha) \\ \theta_\beta &= g(z_\alpha)\theta_\alpha, \end{aligned} \tag{8.75}$$

for an open cover $\{U_\alpha\}_{\alpha\in A}$ of M and local complex coordinates $(z_\alpha, \theta_\alpha)$ on U_α. An odd deformation of J produces a complex structure with additional terms ψ, η of the form (8.72). The collection $\{\psi_{\alpha\beta}\}$ defines an element $\tilde{v}$ of $H^1(M, K^{-1}\otimes L)$, and likewise $\{\eta_{\alpha\beta}\}$ defines an element $\tilde{w}$ of $H^1(M, L^{-1})$. As shown in the preceding discussion, the map

$$(\tilde{v}, \tilde{w}) \mapsto (\alpha\tilde{v}, \alpha^{-1}\tilde{w}) \tag{8.76}$$

alters the cocycle, but does not change the complex structure described by it. Since $H^1(M, K^{-1}\otimes L) \cong H^0(M, K^2 \otimes L^{-1})$ and $H^1(M, L^{-1}) \cong H^0(M, K\otimes L)$, this proves the claim. $\square$

This unpleasant fact will prevent $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ from being an effective parametrization of the deformations of marked complex supersurfaces. If one tries to quotient out this $\mathbb{C}^\times$-action, the result cannot be a supermanifold anymore. We will not pursue this problem further in this work. However, the situation becomes a bit better in the case of a $\mathfrak{k}^L(1|1)$-surface, i.e., a super Riemann surface.

Theorem 8.1.7. *Let $\mathcal{M}$ be a $\mathfrak{k}^L(1|1)$-surface. Then the $\mathbb{C}^\times$-action (8.74) induces a $\mathbb{Z}_2$-action on the odd deformations of the $\mathfrak{k}^L(1|1)$-structure given by*

$$\begin{array}{rcl} \mathbb{Z}_2 \times H^0(M, K^2\otimes L^{-1}) & \to & H^0(M, K^2 \otimes L^{-1}) \\ (n, v) & \mapsto & nv. \end{array} \tag{8.77}$$

The deformations v and nv describe the same $\mathfrak{k}^L(1|1)$-structure.

Proof. Let

$$\begin{array}{rcl} z_\beta & = & f(z_\alpha) \\ \theta_\beta & = & g(z_\alpha)\theta_\alpha, \end{array} \tag{8.78}$$

be the cocycle defining the $\mathfrak{k}^L(1|1)$-structure on $\mathcal{M}$. An odd deformation brings it into the form (8.72), but the requirement that it remain a $\mathfrak{k}^L(1|1)$-structure restricts the transition functions. We have to make sure that $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ is preserved up to an invertible factor, but (8.72) entails

$$D_\alpha = (D_\alpha\theta_\beta)D_\beta + (D_\alpha z_\beta - \theta_\beta D_\alpha \theta_\beta)D_\beta^2. \tag{8.79}$$

So we require $D_\alpha z_\beta = \theta_\beta D_\alpha\theta_\beta$, which is equivalent to

$$\psi_{\alpha\beta} = g_{\alpha\beta}\eta_{\alpha\beta} \tag{8.80}$$

$$g_{\alpha\beta}^2 = f'_{\alpha\beta} + \psi_{\alpha\beta}\psi'_{\alpha\beta}. \tag{8.81}$$

The map $\theta \mapsto \mathrm{e}^c\theta$ is only an automorphism of the $\mathfrak{k}^L(1|1)$-structure if (8.80) and (8.81) remain satisfied. This enforces

$$\mathrm{e}^c = \pm 1. \tag{8.82}$$

This means that the odd deformations of a $\mathfrak{k}^L(1|1)$-structure are effective up this $\mathbb{Z}_2$ ambiguity. $\square$

So the space $\mathcal{T}^g_{\mathfrak{k}^L(1|1)}$ that we will construct below cannot be the base of a universal family either. In this case, however, one could define a "superorbifold" (as is done, e.g., in [LR88]) by quotienting out the residual automorphisms. We will leave the investigation of the resulting superorbifold for later work and content ourselves with the space $\mathcal{T}^g_{\mathfrak{k}^L(1|1)}$.

8.2 Earle's family of Jacobians over Teichmüller space

The Teichmüller space of complex compact closed supersurfaces of dimension $1|1$ will turn out not to be a supermanifold because of the presence of nontrivial infinitesimal automorphisms. Its underlying space, however, is a complex manifold, namely the fiber space $J(V_g) \to \mathcal{T}_g$ of Jacobian varieties over (ordinary) Teichmüller space $\mathcal{T}_g$. The space $J(V_g)$ was introduced by Earle [Ear78] along with an embedding of the universal Teichmüller curve into it. The universal Teichmüller curve is universal among all embeddings of families of Riemann surfaces into families of compact complex manifolds over the same base. We will briefly review this construction in the next section, before proceeding to construct $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$.

8.2.1 The universal Teichmüller curve

Let $\mathcal{T}_g$, $g \geq 2$, be the Teichmüller space of compact Riemann surfaces of genus g, and let Γ be the fundamental group of a smooth surface of genus g. Then the Bers fiber space [Ber73] $F_g \subset \mathcal{T}_g \times \mathbb{C}$ is a subspace of $\mathcal{T}_g \times \mathbb{C}$ with the following properties:

1. Γ acts freely, properly discontinuously and fiberwise on F_g as a group of biholomorphic maps
$$\gamma(t, z) = (t, \gamma^t(z)) \qquad \forall \gamma \in \Gamma, (t, z) \in F_g, \tag{8.83}$$
such that for every $t \in \mathcal{T}_g$, γ^t is a Möbius transformation,

2. the fiber $U(t)$ of F_g over t is simply connected, and $U(t)/\Gamma$ is the Riemann surface represented by t,

3. the projection map $\mathcal{T}_g \times \mathbb{C} \to \mathcal{T}_g$ descends to a well-defined holomorphic map $\pi_0 : V_g := F_g/\Gamma \to \mathcal{T}_g$ defining a holomorphic family of Riemann surfaces.

The family $\pi_0 : V_g \to \mathcal{T}_g$ is called the universal Teichmüller curve. One can express the action of Γ on F_g by the choice of t-dependent elements of $PSL(2, \mathbb{C})$. These matrices will vary holomorphically with t, since Γ is supposed to act biholomorphically on the total space F_g. Only loxodromic[1] elements of $PSL(2, \mathbb{C})$ will occur, since otherwise, the quotients would not be compact.

[1] A Möbius transformation is called loxodromic if it is conjugate in $PSL(2, \mathbb{C})$ to the matrix $M = \mathrm{diag}(\lambda, \lambda^{-1})$ with $|\lambda| \neq 1$. Hyperbolic transformations are a special case of loxodromic ones, namely those for which $\mathrm{tr}(M^2) = \lambda^2 + \lambda^{-2} > 4$.

By the Riemann mapping theorem, each $U(t)$ is biholomorphic to the upper half-plane $\mathbb{H}$. This is just the statement of the uniformization theorem: each compact Riemann surface of genus $g \geq 2$ has $\mathbb{H}$ as its universal covering space. One can then describe the action of Γ as the action of a finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}) = PSL(2,\mathbb{R})$. It is, however, impossible to uniformize all fibers of F_g simultaneously by $\mathbb{H}$ if one wants to keep the action of Γ holomorphic in t.

8.2.2 The basis of normalized differentials

A set of $2g$ canonical generators for Γ is a set $\{A_1, \dots, A_g, B_1, \dots, B_g\} \subset \Gamma$ such that

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1, \tag{8.84}$$

holds, and all other relations in Γ are consequences of this single one. On the quotient $\Sigma_t = U(t)/\Gamma$, these generators are represented by nontrivial loops on the Riemann surface which generate the fundamental group $\pi_1(\Sigma_t)$. These loops have intersection numbers

$$A_j \cdot A_k = B_j \cdot B_k = 0, \qquad A_j \cdot B_k = \delta_{jk} \qquad 1 \leq j,k \leq g \tag{8.85}$$

(cf., e.g., [Jos06] for a geometric treatment of the intersection pairing). Bers [Ber61] has shown that one can construct a family of g linearly independent holomorphic functions $\alpha_j(t,z)$ on F_g which satisfy

$$\alpha_j(t,z) = \alpha_j(\gamma(t,z)) \frac{\partial \gamma}{\partial z}(t,z) \qquad \forall \gamma \in \Gamma, \tag{8.86}$$

and

$$\int_z^{A_k^t(z)} \alpha_j(t,w)dw = \delta_{jk} \qquad \forall t \in \mathcal{T}_g, z \in U(t), 1 \leq j,k \leq g. \tag{8.87}$$

The integral in (8.87) may be computed along any path in $U(t)$. Equation (8.86) just means that the functions α_j will descend to holomorphic 1-forms on the quotient Σ_t, while equation (8.87) states that these 1-forms are the basis of $\Omega^{1,0}(\Sigma_t)$ dual to the loops $A_1, \dots, A_g$. It is a classical fact that these requirements determine the functions α_j uniquely [Jos06]. We can then calculate the Riemann period matrix associated with our chosen basis of loops $A_1, \dots, B_g$:

$$\tau_{ij}(t) := \int_z^{B_i^t(z)} \alpha_j(t,w)dw, \tag{8.88}$$

where z is just any point in $U(t)$. Since both B_i and α_j are holomorphic with respect to t, the entries of the period matrix are as well.

8.2.3 The family of Jacobians

It is a classical fact that the g column vectors τ_i, $1 \leq i \leq g$ of the period matrix and the g standard basis vectors $e_1, \dots, e_g$ of $\mathbb{C}^g$ are linearly independent over the reals [Jos06]. They form a lattice subgroup $(\mathbb{1}, \tau)$ in $\mathbb{C}^g$, and the Jacobian variety of a Riemann surface Σ is a g-dimensional torus defined as

$$\mathrm{Jac}(\Sigma) := \mathbb{C}^g/(\mathbb{1}, \tau). \tag{8.89}$$

It is Earle's result [Ear78] that this construction can be carried out holomorphically on the entire universal curve.

Theorem 8.2.1 (Earle)**.** *The $\mathbb{Z}^g \times \mathbb{Z}^g$-action on $\mathcal{T}_g \times \mathbb{C}^g$ given by*

$$\begin{aligned} \Lambda : \mathbb{Z}^g \times \mathbb{Z}^g \times (\mathcal{T}_g \times \mathbb{C}) &\to \mathcal{T}_g \times \mathbb{C} && (8.90)\\ (m, n, t, z) &\mapsto (t, z + m + \tau(t)n) && (8.91) \end{aligned}$$

is a free, properly discontinuous action by biholomorphisms. The projection $\mathcal{T}_g \times \mathbb{C}^g \to \mathcal{T}_g$ descends to a holomorphic family $\pi : J(V_g) := (\mathcal{T}_g \times \mathbb{C}^g)/\Lambda \to \mathcal{T}_g$ of complex tori. The family $J(V_g)$ is a topologically trivial fibration, and each fiber $\pi^{-1}(t)$ is canonically isomorphic to $\mathrm{Jac}(\Sigma_t)$, the Jacobian of the corresponding Riemann surface.

8.2.4 The Picard variety

The isomorphism classes of holomorphic line bundles on a Riemann surface Σ form an abelian group, the Picard group $\mathrm{Pic}(\Sigma)$, with group operation the tensor product of bundles. This group can be written in a variety of ways. First of all, one can view $\mathrm{Pic}(\Sigma)$ as the group of linear equivalence classes of divisors on Σ. From this point of view, the Picard variety inherits naturally the degree map of divisors

$$\deg : \mathrm{Pic}(\Sigma) \to \mathbb{Z}, \tag{8.92}$$

which is easily seen to be a homomorphism of abelian groups. On the other hand, the set of isomorphism classes of line bundles is naturally isomorphic to the first Čech cohomology group $H^1(\Sigma, \mathcal{O}^*)$, with $\mathcal{O}^*$ the sheaf of invertible holomorphic functions on Σ. The two pictures can be connected using the exponential sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0, \tag{8.93}$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf with values in $\mathbb{Z}$ and exp is the map which sends each germ f to $\exp(2\pi i f)$. This sequence induces a long exact sequence in cohomology, of which a part reads

$$\ldots \to H^1(\Sigma, \underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma, \mathcal{O}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \xrightarrow{c} H^2(\Sigma, \underline{\mathbb{Z}}) \to \ldots \tag{8.94}$$

Since Σ is connected, we have $H^2(\Sigma, \underline{\mathbb{Z}}) \cong \mathbb{Z}$. The connecting homomorphism c sends a line bundle to its Chern class, so in our case it is simply the degree map. Hence, we can conclude that for the line bundles of degree zero, we have

$$\mathrm{Pic}^0(\Sigma) \cong \exp(H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \underline{\mathbb{Z}})). \tag{8.95}$$

Line bundles of degree zero form a normal subgroup of $\mathrm{Pic}(\Sigma)$ which can be identified with the Jacobian via the Abel map. Choose $p_0 \in \Sigma$. Then the Abel map is given by

$$\begin{aligned} \mathcal{A}_{p_0} : \Sigma &\rightarrow \mathrm{Jac}(\Sigma) && (8.96)\\ \mathcal{A}_{p_0}(p) &= \left(\int_{p_0}^{p} \alpha_1, \ldots, \int_{p_0}^{p} \alpha_g \right) && (8.97) \end{aligned}$$

This mapping depends on the base point p_0 and on our choice of a basis $A_1, \ldots, B_g$ (see above) for the first homology of Σ.[2] The 1-forms being integrated over in the Abel map are those dual to the A_i. The Abel map is an embedding and induces a homomorphism

$$\mathcal{A}_{p_0} : \mathrm{Div}(\Sigma) \rightarrow \mathrm{Jac}(\Sigma), \tag{8.98}$$

where Div is the abelian group of divisors on Σ. For divisors of degree 0, the Abel map is independent of the base point. Moreover, Abel's theorem asserts that

$$\mathcal{A}_{p_0}(D) = 0 \quad \Leftrightarrow \quad D \in \mathrm{PDiv}(\Sigma), \tag{8.99}$$

where $\mathrm{PDiv}(\Sigma)$ is the space of principal divisors (those which can occur as the divisors of meromorphic functions). This shows that one can let $\mathcal{A}_{p_0}$ descend to a map on the group of linear equivalence classes of divisors, and that it is injective there. In addition, the Jacobi inversion theorem states that $\mathcal{A}_{p_0}$ is surjective. Therefore, we have the sequence

$$\mathcal{A}_{p_0} : \mathrm{Jac}(\Sigma) \cong (\mathrm{Div}(\Sigma)/\sim) \cong \mathrm{Pic}^0(\Sigma), \tag{8.100}$$

of isomorphisms of abelian varieties, where $\sim$ denotes linear equivalence. Since $\deg : \mathrm{Pic} \rightarrow \mathbb{Z}$ is a homomorphism, we see that the set of line bundles of degree d can be obtained as the coset

$$\mathrm{Pic}^d(\Sigma) \cong L \otimes \mathrm{Pic}^0(\Sigma) \tag{8.101}$$

of any line bundle L of degree d. None of these cosets except for $d = 0$ is a group, but each one is isomorphic as a complex manifold to $\mathrm{Pic}^0(\Sigma) \cong \mathrm{Jac}(\Sigma)$ [Jos06], [GH94].

[2] Actually we had introduced $A_1, \ldots, B_g$ as the generators of $\pi_1(\Sigma)$. But by a theorem of van Kampen (see, e.g., [Jos06]) one has $H_1(\Sigma, \mathbb{Z}) \cong (\pi_1(\Sigma)/[,])$, where $[,]$ denotes the commutator subgroup of $\pi_1(\Sigma)$. Thus H_1 is isomorphic to the free abelian group generated by $A_1 \ldots, B_g$.

8.2.5 The embedding of V_g into $J(V_g)$

With the family $J(V_g) \to \mathcal{T}_g$ at hand, one can ask oneself whether it is possible to extend the Abel embedding $\mathcal{A}_{p_0} : \Sigma \to \mathrm{Jac}(\Sigma)$ to the entire family $J(V_g)$. The problem here is that one cannot choose a point $p_0(t)$ on $\Sigma(t)$ which varies holomorphically with the coordinates t of $\mathcal{T}_g$: the universal curve V_g has no holomorphic sections if $g \geq 3$ [Hub76].

As before, let $A_1, \ldots, B_g$ denote a canonical basis of $H_1(\Sigma)$, and let $\omega_1, \ldots, \omega_g$ be the dual basis of holomorphic 1-forms on Σ. For any base point $p_0 \in \Sigma$, we define the vector $K(p_0)$ of Riemann constants as the point of $\mathrm{Jac}(\Sigma)$ whose j-th component is

$$K_j(p_0) = -\frac{\tau_{jj}}{2} + \sum_{k \neq j} \int\limits_{A_j} \omega_j(w) \int\limits_{p_0}^{w} \omega_k(s), \tag{8.102}$$

where τ is the period matrix (8.88) of Σ. The vector K plays an important role in the study of theta functions of Riemann surfaces, as well as for spin structures, as will be shown below. It also plays a key role in the construction of an embedding $V_g \to J(V_g)$ as Earle has shown.

Theorem 8.2.2 (Earle [Ear78]). *The map $\eta : F_g \to \mathbb{C}^g$ defined by*

$$\eta_j(t, z) := \frac{1}{1-g} \left(\sum_{k \neq j} \int\limits_{z}^{A_j(z)} \alpha_j(w) \left(\int\limits_{z}^{w} \alpha_k(u) du \right) dw - \frac{\tau_{jj}}{2} \right) \tag{8.103}$$

is holomorphic and satisfies

$$\frac{\partial \eta_j}{\partial z} = \alpha_j, \qquad \eta(A_k(t,z)) = \eta(t,z) + e_k, \qquad \eta(B_k(t,z)) = \eta(t,z) + \tau(t) e_k \tag{8.104}$$

for $j, k = 1, \ldots, g$. Here e_k is the k-th unit vector of $\mathbb{C}^g$.

The map η just assigns to every point (t, z) the (normalized) vector of Riemann constants $\frac{1}{1-g} K(t,z)$ relative to this point. The properties (8.104) show that η descends to a map $\psi : V_g \to J(V_g)$, the desired embedding.

Theorem 8.2.3 (Earle [Ear78]). *Let $\pi : J(V_g) \to \mathcal{T}_g$ and $\pi_0 : V_g \to \mathcal{T}_g$ be the family of Jacobians and the universal Teichmüller curve, respectively. Then the map $\psi : V_g \to J(V_g)$ defined by the map η above is a holomorphic embedding such that $\pi_0 = \pi \circ \psi$.*

The Abel map with base point p_0 can be viewed as a map $\Sigma \to \mathrm{Div}^0(\Sigma)$ which assigns to each $p \in \Sigma$ the divisor class $[p - p_0]$. Conversely, if we have any divisor D of degree 1 on Σ, then the map $p \mapsto [p - D]$ is an embedding $\Sigma \to \mathrm{Jac}(\Sigma)$, and all translates of the Abel map can be obtained this way. Therefore, Thm. 8.2.3 defines a divisor class $[D(t)]$ on every fiber $\Sigma(t)$ of V_g, and this class varies holomorphically in t. The class $[(g-1)D(t)]$ is the vector of Riemann constants, by construction. Moreover, a class of degree 1 and the Jacobian variety are sufficient to reproduce all equivalence classes of divisors, and therefore the entire Picard variety.

8.2.6 Spin bundles on Riemann surfaces

It was already mentioned in Chapter 4 that a spin bundle on a Riemann surface Σ is just a square root of the canonical bundle K, and that the number of inequivalent spin bundles on Σ is given by $\left|H^1(\Sigma, \mathbb{Z}_2)\right| = 2^{2g}$. This already makes it clear that the divisors belonging to spin bundles form a discrete subset of $\mathrm{Jac}(\Sigma)$, and that there is no nontrivial way to continuously deform a spin bundle in such a way that it remains a spin bundle throughout the deformation.

The distribution of spin bundle divisors on $\mathrm{Jac}(\Sigma)$ can be easily determined with the help of the following well-known theorem.

Theorem 8.2.4. *Let $C \in \mathrm{Jac}(\Sigma)$ denote the divisor class of the canonical bundle, and let $K(p_0)$ be the vector of Riemann constants relative to the point $p_0 \in \Sigma$. Then*

$$2K(p_0) = C \qquad \textit{for all } p_0 \in \Sigma. \tag{8.105}$$

Since for a spin bundle S, we have $K = S \otimes S$, we must also have $2[S] = C$ for the divisor class $[S]$ of S. Therefore the set of points of $\mathrm{Jac}(\Sigma)$ which belong to divisors of spin bundles is a translation of the half-period lattice

$$\Delta = \alpha + \tau \cdot \beta, \qquad \alpha, \beta \in \frac{1}{2}\mathbb{Z}^g \tag{8.106}$$

(τ the period matrix) by the vector of Riemann constants. The elements of Δ are also called theta characteristics or points of 2-torsion of $\mathrm{Jac}(\Sigma)$. One distinguishes odd and even spin structures by the parity of the integer $4(\alpha \cdot \beta) = 4 \sum \alpha_k \beta_k$.

The number of holomorphic sections of a spin bundle on a Riemann surface is, in general, difficult to determine. The Riemann-Roch theorem is not of much use for spin bundles: inserting a square root S of K, one can only conclude that $\deg(S) = g - 1$, which was clear anyway. Moreover, spin bundles lie in the "non-topological" range (8.69), and one can indeed show that the number $h^0(\Sigma, S)$ depends on the particular complex structure Σ, i.e., it can jump when one runs along a path in Teichmüller space [ACGH85]. At least for odd spin structures, a holomorphic section is guaranteed since one can show that $h^0(\Sigma, S) \equiv$ parity of S mod 2.

Earle's embedding $V_g \to J(V_g)$ makes it possible to keep track of a particular spin structure along the entire universal curve, since the vector of Riemann constants as the reference point is given as a global holomorphic section of $J(V_g)$. It therefore makes sense to compare spin structures on different fibers of the universal curve. One says that two spin structures on two respective fibers $\Sigma_1, \Sigma_2 \subset V_g$ are equal, if they correspond to the same theta characteristic relative to the class $[(g-1)D(t)]$ which represents the vector of Riemann constants on each fiber.

Proposition 8.2.5. *Let $\mathcal{S} \to V_g$ be a holomorphic line bundle over the universal curve V_g such that each restriction $\mathcal{S}(t)$ to a fiber $\Sigma(t) = \pi^{-1}(t)$ of V_g is a spin bundle on $\Sigma(t)$. Then each $\mathcal{S}(t)$ has the same spin structure.*

Proof. Earle's embedding gives us a global section $[(g-1)D(t)]$ of $J(V_g)$ which can be interpreted as the image under the Abel map of the divisor of degree $g-1$ corresponding to the vector of Riemann constants. Furthermore, the half-period lattice (8.106) is given as $\Delta(t)$ on every fiber and varies holomorphically with t, since the period matrix τ is holomorphic in t. Therefore, we can globally define the translated lattice $[(g-1)D(t)] + \Delta(t)$ on V_g, which consists of a set of 2^{2g} holomorphic sections of $J(V_g)$ describing the spin structures of $\Sigma(t)$. It is clear that the divisor class of $\mathcal{S}(t)$ has to vary holomorphically in t if $\mathcal{S}$ is holomorphic in t. Therefore, this class corresponds to one of the sections of $J(V_g)$ defined by $[(g-1)D(t)] + \Delta(t)$. □

The same does not hold anymore if one allows diffeomorphisms of Σ which are not homotopic to id_Σ. Then the spin structures can get interchanged, but their parity remains preserved [ACGH85]. The moduli space of spin curves therefore consists of two connected components [Cor89]. But in this work, we will confine ourselves to the construction of Teichmüller spaces, where these problems do not arise and Prop. 8.2.5 holds. For a very detailed account on the moduli of spin curves and theta characteristics see [Cor89], [ACGH85].

8.2.7 The Teichmüller space of pairs of Riemann surfaces and line bundles

The discussion of the previous section can be summed up by saying that the Jacobian $\mathrm{Jac}(\Sigma)$ of a given Riemann surface Σ can be viewed as the moduli space of line bundles of some fixed degree d on Σ: it effectively parametrizes all isomorphism classes. The existence of Earle's family of Jacobians allows one to extend this to pairs of complex structures and holomorphic line bundles.

Let M be a given smooth compact closed oriented surface M together with a smooth orientable vector bundle $p : E \to M$ of rank 2 and degree d. Denote by $\mathcal{C}^{g,d}$ the set of pairs (J_E, J_M) of almost complex structures on E and M, respectively, such that

1. J_E is integrable, and
2. the bundle projection p becomes a holomorphic map with respect to J_E and J_M.

The space of these pairs is obviously the space of pairs (M, L) consisting of a Riemann surface of genus g and a holomorphic line bundle $p : L \to M$ of degree d. On $\mathcal{C}^{g,d}$, we have an action of the group $\mathrm{Aut}_{\mathsf{VBun}}(M, E)$ of automorphisms of E as a smooth vector bundle (see Definition 7.3.2). We recall that this group splits as a semidirect product

$$\mathrm{Aut}_{\mathsf{VBun}}(M, E) = \mathrm{Diff}(M) \ltimes \mathrm{Aut}_{\mathsf{VBun}(M)}(E), \tag{8.107}$$

where $\mathrm{Aut}_{\mathsf{VBun}(M)}(E)$ is the group of automorphisms of E over M, i.e. the subgroup of $\mathrm{Aut}_{\mathsf{VBun}}(M, E)$ whose elements act as the identity on M.

The elements of $\mathrm{Diff}(M)$ act on $\mathcal{C}^{g,d}$ by pullback of the entire bundle along with J_E and J_M. The elements of $\mathrm{Aut}_{\mathsf{VBun}(M)}(E)$ preserve J_M and act only by real linear automorphisms on the fibers of E. Quotienting out the acion of $\mathrm{Aut}_{\mathsf{VBun}}(M,E)$ would produce the moduli space of pairs (M,L), which is too ambitious for our purposes. Let us instead restrict the underlying diffeomorphisms to the group $\mathrm{Diff}_0(M)$, i.e. to those homotopic to the identity. This gives us the group $\mathrm{Aut}_{\mathsf{VBun}}(M,E)_0$. The quotient of $\mathcal{C}^{g,d}$ by the action of this group is the Teichmüller space of pairs (M,L).

Proposition 8.2.6. *The family $J(V_g) \to \mathcal{T}_g$ of Jacobians is the Teichmüller space of pairs (M,L) consisting of a Riemann surface of genus g and a holomorphic line bundle $L \to M$ of degree d.*

Proof. Given the space $\mathcal{C}^{g,d}$, we can quotient out the action of $\mathrm{Aut}_{\mathsf{VBun}(M)}(E)$ to obtain the space $J(\mathcal{C}^{g,d})$ of pairs $(M,[L])$ of complex structures on M and equivalence classes of holomorphic line bundles of degree d on M. This space then carries an action of

$$\mathrm{Aut}_{\mathsf{VBun}}(M,E)_0/\mathrm{Aut}_{\mathsf{VBun}(M)}(E) \cong \mathrm{Diff}_0(M). \tag{8.108}$$

The quotienting out of this action produces the space of pairs $([M]_{\mathrm{Diff}_0(M)},[L])$ of $\mathrm{Diff}_0(M)$-orbits of complex structures on M and isomorphism classes of holomorphic line bundles. By Earle's result, this space can be given a natural complex structure, namely that of the family $J(V_g)$. $\square$

8.3 The super Teichmüller space $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$

8.3.1 Versal, semiuniversal and universal families

The Teichmüller space $\mathcal{T}_g$ of Riemann surfaces of genus g is the base of a universal family of marked Riemann surfaces. This means that any other complex analytic family of marked surfaces can be obtained by pullback from it *in a unique way.* The supermanifold $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ that will be constructed in this section is, however, not the base of a universal family of complex $1|1$-dimensional supermanifolds. This is due to the existence of automorphisms of these structures which cannot be divided out without destroying the supermanifold structure of the base (cf. Thm. 8.1.6). Before proceeding to the construction of $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$, we want to introduce some basic terminology of deformation theory and argue that $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ can still be regarded as the super Teichmüller space of $\mathfrak{vect}^L(1|1)$-structures. For the sake of clarity, we look at ordinary complex manifolds first. For detailed accounts, consult, e.g., [Man04], [Kod86], [SU02].

Let $\pi : M \to B$ be a complex analytic family of compact complex manifolds. This means that π is a proper holomorphic map and $\pi_* : T_pM \to T_{\pi(p)}B$ is surjective at every $p \in M$. For every $b \in B$, such a family is also called a *deformation*

of the fiber $M_b := \pi^{-1}(b)$. Assume for simplicity that B is a polydisc (since we are only interested in local properties around $b \in B$, this is not a restriction). We denote by $\Gamma_{hol}(B, TB^{1,0})$ the holomorphic sections of the holomorphic tangent bundle of the base and by $\Gamma(M, TM^{1,0})$ the smooth holomorphic vector fields on M. The *Kodaira-Spencer map* $\mathcal{K}S_\pi$ is defined as

$$\begin{aligned} \mathcal{KS}_\pi : \Gamma_{hol}(B, TB^{1,0}) &\rightarrow H^1(M, T(M/B)^{1,0}) && (8.109) \\ \gamma &\mapsto [\bar{\partial}\eta], && (8.110) \end{aligned}$$

where $\eta \in \Gamma(M, TM^{1,0})$ is any vector field such that $\pi_*\eta = \gamma$ and $T(M/B) := \ker \pi_*$ denotes the relative tangent bundle. It is easy to show that this map is well-defined, i.e., does not depend on the chosen lift η of γ. The map $\mathcal{KS}_\pi$ induces a map

$$KS_{\pi,b} : T_b B^{1,0} \rightarrow H^1(M_b, TM_b^{1,0}) \tag{8.111}$$

which one obtains from the map that $\mathcal{KS}_\pi$ induces on the stalks by dividing out the ideal of germs of vector fields vanishing at b. This map is also called the Kodaira-Spencer map at b. The Kodaira-Spencer mapping plays a central role in deformation theory. Originally it was invented to study families of complex manifolds, but one can already see from the above definitions that it is possible to apply it in a variety of other contexts. Vaintrob [Vai88a] has developed a Kodaira-Spencer theory for complex superspaces and showed its usefulness in several examples. One can interpret $KS_{\pi,b}$ as a means to measure how much the complex structure on the fibers of the family varies in a neighbourhood of b. Geometrically speaking, the cohomology group $H^1(M_b, TM_b^{1,0})$ describes the possible variations of the transition functions of M_b which cannot be induced by mere coordinate changes, i.e., those which can define a new complex structure which is not biholomorphic to the one on M_b.

The map $KS_{\pi,b}$ is also used to classify families in the sense of "how well" they parametrize the objects in question (in the above definition: compact complex manifolds). For this we need to have a look at the behaviour of KS_π under pullback. Let $\phi : C \rightarrow B$ be a holomorphic map and let $b = \phi(c)$. We obtain a pullback family $M \times_B C$ whose projections we denote as

$$\begin{array}{ccc} M \times_B C & \xrightarrow{\hat{\phi}} & M \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\phi} & B \end{array} \quad . \tag{8.112}$$

Then one finds (see, e.g., [Man04]):

Theorem 8.3.1. *We have*

$$KS_{\hat{\pi},c} = KS_{\pi,b} \circ \phi_* : T_c C \rightarrow H^1(M_b, TM_b^{1,0}). \tag{8.113}$$

Definition 8.3.2. *Let $\pi : M \to B$ be a complex analytic family of compact complex manifolds and let $b \in B$ be given together with the Kodaira-Spencer map $KS_{\pi,b} : T_bB \to H^1(M_b, TM_b^{1,0})$. Then the family π is called*

- versal *at b if $KS_{\pi,b}$ is surjective, and for every family $p : N \to C$ around $c \in C$ there exists a morphism $\phi : C \to B$ with $\phi(c) = b$ such that the pullback $M \times_B C$ is isomorphic to $p : N \to C$,*

- semiuniversal *at b if it is versal and $KS_{\pi,b}$ is a bijection, and*

- universal *at b if $KS_{\pi,b}$ is a bijection and if for every family $p : N \to C$ around $c \in C$ there exists a unique morphism $\phi : C \to B$ with $\phi(c) = b$ such that the pullback $M \times_B C$ is isomorphic to $p : N \to C$.*

Semiuniversal families are also called *Kuranishi* families. The interpretation of this classification is quite clear: a universal family parametrizes objects "optimally", in the sense that it contains all possible deformations of the fiber M_b and that every family which contains a fiber isomorphic to M_b can be pulled back from the universal one in a unique way. The Teichmüller space of Riemann surfaces is the base of such a universal family for marked Riemann surfaces. Teichmüller space does not, however, universally parametrize unmarked families of Riemann surfaces. Without a marking, a Riemann surface may possess an additional discrete group of automorphisms, and in this case there is no unique way to obtain a family containing it as a pullback from the family over Teichmüller space. But it is still semiuniversal: it is possible to pull back any family from it, albeit non-uniquely, and the tangent space at Teichmüller space can naturally be identified with $H^1(M, TM^{1,0}) \cong H^1(M, K) \cong H^0(M, K^2)$ (if M is a Riemann surface). The moduli space of Riemann surfaces is the quotient of Teichmüller space by the mapping class group, whose elements represent these additional possible automorphisms. It is universal in the sense that it parametrizes the deformations "optimally", but the price is that it is not a manifold anymore. Therefore, a universal family of Riemann surfaces of genus $g \geq 2$ does not exist (at least not in the category of complex manifolds, but in the larger category of stacks it does). Even semiuniversal families do not, in general, exist. A versal family still contains all deformations of the fiber M_b, but may contain redundancies. Versal families are also called *complete*.

8.3.2 Construction of $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$

The result of Section 8.2.7 is already almost the answer for the underlying Teichmüller space of $\mathfrak{vect}^L(1|1)$-structures. A smooth supermanifold of dimension $2|2$ is, since smooth supermanifolds are always split, isomorphic to a supermanifold of the form $\mathcal{M} = (M, \wedge E)$, where E is a smooth vector bundle of rank 2 over the smooth underlying manifold M. We assume that M is compact, closed and orientable, and also that E is orientable. The genus $g \geq 2$ of M and the degree

d of E are invariants of $\mathcal{M}$, i.e., they are preserved by superdiffeomorphisms. A complex supermanifold of dimension $1|1$, as was shown in Prop. 4.2.1, is always of the form $(M, \mathcal{O}_M \oplus \Gamma(L))$, where M is a Riemann surface and L is a holomorphic line bundle over M. Since there is only one odd coordinate available, it always remains in this exterior bundle form. The underlying smooth supermanifold, however, may deviate from the form $(M, \wedge^\bullet E)$: its transition functions can contain terms proportional to $\theta_1\theta_2$. So there seems to be a mismatch: when we switch back from (M, L) to the underlying smooth bundle $(M, \wedge^\bullet E)$, we always obtain the canonical bundle form $\mathcal{M}^{split}$, but never the nilpotent corrections proportional to $\theta_1\theta_2$ that may occur in the smooth supermanifold $\mathcal{M}$.

The solution to this riddle is that a real form on $\mathcal{M}$, i.e., an involution of the complex supermanifold whose fixed point locus is again the underlying real analytic supermanifold [DM99], need not express the even complex coordinate z solely as a function of the even real coordinates x, y. It only must produce an even real superfunction, which can involve terms proportional to $\theta_1\theta_2$ as well.

Theorem 8.3.3. *The family $\pi : J(V_g) \to \mathcal{T}_g$ is the underlying manifold of $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$, i.e., each point of $J(V_g)$ represents exactly one $\widehat{\mathcal{SD}}_0(\mathcal{M})(\mathbb{R})$-orbit in $\mathcal{C}(\mathcal{M})(\mathbb{R})$.*

Proof. Let the associated exterior bundle of $\mathcal{M}$ be denoted as $\mathcal{M}^{split} = (M, \wedge^\bullet E)$, where E is a smooth, orientable vector bundle of rank 2 and degree d. By Thm. 7.3.4, the group $\widehat{\mathcal{SD}}_0(\mathcal{M})(\mathbb{R})$ can be written as

$$\begin{aligned}
\widehat{\mathcal{SD}}_0(\mathcal{M})(\mathbb{R}) &\cong (\mathrm{Diff}_0(M) \ltimes \mathrm{Aut}_{\mathsf{VBun}(M)}(E)) \ltimes N_{\mathcal{M}} && (8.114)\\
&= \mathrm{Aut}_{\mathsf{VBun}}(M, E)_0 \ltimes N_{\mathcal{M}} && (8.115)\\
&= \mathrm{Aut}(\mathcal{M}^{split})_0 \ltimes N_{\mathcal{M}}. && (8.116)
\end{aligned}$$

Therefore, the theorem will be proven if we can divide out the action of $N_{\mathcal{M}}$. Let J be an integrable almost complex structure on $\mathcal{M}$. The group $N_{\mathcal{M}}$ is generated by vector fields of order $j \geq 2$ in the odd variables (see Section 7.3). So if $n \in N_{\mathcal{M}}$ is such a unipotent diffeomorphism and $(x, y, \theta_1, \theta_2)$ is a local smooth coordinate system on $\mathcal{M}$, the action of n can locally be expressed by the sheaf map

$$n = \exp(\mathcal{X}) = (\mathbb{1} + g(x,y)\theta_1\theta_2\frac{\partial}{\partial x} + h(x,y)\theta_1\theta_2\frac{\partial}{\partial y}), \tag{8.117}$$

where g and h are ordinary smooth functions on the base M. The action of n on J is therefore

$$n^*J = J + \theta_1\theta_2 L_X J, \tag{8.118}$$

where $X = g(x,y)\frac{\partial}{\partial x} + h(x,y)\frac{\partial}{\partial y}$. If $L_X J \equiv 0$ were possible for some $X \neq 0$, the action would not be free. But this cannot happen: by Prop. 6.2.1, this would imply that X is holomorphic with respect to J, i.e., it would be a holomorphic vector field with holomorphic coefficients on the underlying Riemann surface. But for genus $g \geq 2$, such holomorphic sections of $TM^{1,0}$ do not exist. Therefore,

$N_{\mathcal{M}}$ acts freely on $\mathcal{C}(\mathcal{M})(\mathbb{R})$. Since the action is also infinitesimal, i.e., by Lie derivatives, its freeness is sufficient for allowing us to divide it out. This is done by quotienting $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}/\mathcal{J}^2$ ($\mathcal{J}$ is the nilpotent ideal of $\mathcal{O}_{\mathcal{M}}$), and by reducing all modules over $\mathcal{O}_{\mathcal{M}}$ correspondingly. This eliminates all tensor fields whose coefficient functions are of degree greater than one in the odd variables. As was shown in Section 2.2.6, this reduces $\mathcal{M}$ to $\mathcal{M}^{split} = (M, \wedge^\bullet E)$. The remaining integrable almost complex structures are complex structures on the vector bundle (M, E), which turn it into a holomorphic line bundle over a Riemann surface, i.e., we have

$$\mathcal{C}(\mathcal{M})(\mathbb{R})/N_{\mathcal{M}} \cong \mathcal{C}^{g,d}, \tag{8.119}$$

where $\mathcal{C}^{g,d}$ is the set of all pairs of Riemann surfaces and holomorphic line bundles defined in the previous section. But then Prop. 8.2.6 implies that the remaining action of $\mathrm{Aut}_{\mathsf{VBun}}(M, E)_0$ on $\mathcal{C}^{g,d}$ can be quotiented out and one obtains $J(V_g)$ as the resulting underlying Teichmüller space. $\square$

This Theorem implies that once one has fixed a marking on the underlying Riemann surface M of a complex $1|1$-dimensional supermanifold $\mathcal{M}$, there exists a universal family for the the purely even deformations of the complex structure of $\mathcal{M}$. This means that among all families of complex $1|1$-dimensional supermanifolds over purely even bases, the one over $J(V_g)$ is universal. On the other hand, it is clear from Thm. 8.1.6 that it will be impossible to construct a supermanifold which is the base of a universal family of marked complex $1|1$-dimensional supermanifolds. The supermanifold $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$ that we will construct below is only the base of a semiuniversal family.

Let $\pi : J(V_g) \to \mathcal{T}_g$ be the family of Jacobians over (ordinary) Teichmüller space. Then each point $p \in J(V_g)$ denotes a pair (M, L) consisting of an isomorphism class of complex structures and an isomorphism class of line bundles which are holomorphic with respect to this complex structure. Let the degree of L be d with $d > 2g-2$. Then we can construct a holomorphic vector bundle $p : H \to J(V_g)$ by assigning to each $p \in J(V_g)$ the space of holomorphic sections of the bundle L.

We employ this construction to define a specific bundle $p : \mathcal{H} \to J(V_g)$ whose fiber at a point (M, L) is the space $H^0(M, K^2 \otimes L^{-1}) \oplus H^0(M, K \otimes L)$ where the degree of L lies in the range $0 \le d \le 2g-2$. This is a holomorphic vector bundle of rank $4g-4$ over $J(V_g)$.

Claim 8.3.4. *Let J be an integrable almost complex structure on a smooth closed oriented supersurface $\mathcal{M}$ of dimension $2|2$. Then $\mathcal{M}$ is equivalent to a Riemann surface M and a holomorphic line bundle $L \to M$ by Prop. 4.2.1. Assume that the genus of M is $g \ge 2$ and that $0 < \deg(L) < 2g-2$. Then the supermanifold $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)} := (J(V_g), \wedge^\bullet \mathcal{H})$ parametrizes a semiuniversal family of complex $1|1$-dimensional supermanifolds.*

The genus g and the degree d remain, of course, constant in this family: they are supersmooth invariants of $\mathcal{M}$. It is clear that the condition that the Kodaira-Spencer map be an isomorphism at every point is satisfied by construction for the family parametrized by $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$. What has to be shown is that every other family $p : \mathcal{V} \to \mathcal{N}$ of compact closed complex $1|1$-dimensional supermanifolds with genus g and degree d can be obtained as a pullback from the one over $\mathcal{T}^{g,d}_{\mathfrak{vect}^L(1|1)}$. This would require a much deeper analysis of the deformation theory of complex superspaces. Although Vaintrob developed this theory in [Vai88a], there still is quite some work to do to prove this claim. We will leave it for further investigations.

8.4 The Teichmüller space $\mathcal{T}^{g}_{\mathfrak{k}^L(1|1)}$

It was shown in Prop. 4.2.2 that the set of $\mathfrak{k}^L(1|1)$-structures on a given smooth supersurface of dimension $2|2$ can be characterised as the set of pairs (M, S), where M is the underlying smooth surface equipped with a complex structure and S is a spin bundle. Spin bundles have degree $g-1$, so they lie in the range (8.69) of allowed degrees and there are no odd infinitesimal automorphisms. Inserting a spin bundle S as the line bundle in our considerations above yields

$$W = H^0(M, K^2) \oplus H^0(M, K) \oplus \Pi \left(H^0(M, K^{3/2}) \oplus H^0(M, K^{3/2}) \right), \qquad (8.120)$$

as the space of nontrivial deformations. Here, $K^{3/2}$ is a shorthand for $K \otimes S$, so this bundle also depends on the spin bundle S. This is a complex super vector space of dimension $4g-3|4g-4$. But the space W does not describe the true deformations of a $\mathfrak{k}^L(1|1)$-structure, it only describes the deformations of a $\mathfrak{vect}^L(1|1)$-structure whose line bundle L happens to be a spin bundle. These deformations would allow a deformation of the isomorphism class $[L]$, but we have to assure that it always remains a spin bundle, so the class really has to stay fixed. It will turn out that we have to restrict the space W to a certain subspace. In order to do that, we first reinterpret the tangent space V to the slice as a cohomology group, which is the form usually used in deformation theory.

Lemma 8.4.1. *Let $\mathcal{M}$ be a complex supersurface of dimension $1|1$, and let (M, L) be the Riemann surface and line bundle to which $\mathcal{M}$ is equivalent. Then the space V constructed in (8.65) is $H^1(M, \mathcal{TM})$, the first Čech cohomology group of the sheaf $\mathcal{TM}$ considered as a $\mathbb{Z}_2$-graded module over the sheaf $\mathcal{O}_M$ of holomorphic functions on the base manifold M.*

Proof. Since a complex supersurface $\mathcal{M}$ of dimension $1|1$ can always be interpreted as a Riemann surface M with a parity-reversed line bundle L, the tangent sheaf $\mathcal{TM}$ can indeed always be interpreted as a locally free module over $\mathcal{O}_M$ of rank $2|2$. A local basis is given by

$$\frac{\partial}{\partial z}, \quad \theta\frac{\partial}{\partial \theta}, \quad \theta\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \theta}. \qquad (8.121)$$

The first two of these are even, the last two are odd. Interpreting θ as a local section of L and $\frac{\partial}{\partial\theta}$ as a local section of L^{-1} yields the result. $\square$

The interpretation of this result is as follows. Associating to every open set of M the space of sections of $\mathcal{TM}$, one obtains a presheaf with values in the Lie algebra $\mathfrak{vect}^L(1|1)$. This is the algebra of infinitesimal local automorphisms of the supercomplex structure. Its first cohomology describes precisely those variations of the supercomplex structure which cannot be achieved by a global coordinate change, i.e., by a diffeomorphism.

In the case of a $\mathfrak{k}^L(1|1)$-structure, the algebra of local infinitesimal automorphisms is, of course, $\mathfrak{k}^L(1|1) \subset \mathfrak{vect}^L(1|1)$. We must therefore determine the first Čech cohomology of the sheaf of $\mathfrak{k}^L(1|1)$-vector fields as the tangent space to $\mathcal{T}^g_{\mathfrak{k}^L(1|1)}$. Although it is a slight abuse of notation, we will denote the sheaves whose sections are elements of a Lie algebra $\mathfrak{g}$ also as $\mathfrak{g}$. For example, $\mathfrak{k}^L(1|1)$ also denotes the subsheaf of $\mathcal{TM}$ whose local sections form the algebra $\mathfrak{k}^L(1|1)$.

Lemma 8.4.2. *Let $\mathcal{M}$ be a complex supersurface with a $\mathfrak{k}^L(1|1)$-structure, i.e., $\mathcal{M}$ is complex $1|1$-dimensional and it is endowed with a maximally nonintegrable distribution $\mathcal{D}$ of rank $0|1$ (cf. Definition 4.26). Then there exists a direct decomposition of $\mathcal{O}_\mathcal{M}$-modules*

$$\mathcal{TM} = \mathfrak{vect}^L(1|1) = \mathfrak{k}^L(1|1) \oplus \mathcal{D}. \tag{8.122}$$

Proof. The statement is local, so we can work in local complex coordinates z, θ and can assume that $\mathcal{D}$ is generated by $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. From Prop. 4.1.3 we know that every element of $\mathfrak{k}^L(1|1)$ can be obtained by taking an arbitrary function $f \in \mathcal{O}_\mathcal{M}$ and calculating its associated contact vector field

$$K_f = (2 - \theta\frac{\partial}{\partial\theta})(f)\frac{\partial}{\partial z} + (-1)^{p(f)}\frac{\partial f}{\partial\theta}\frac{\partial}{\partial\theta} + \frac{\partial f}{\partial z}\theta\frac{\partial}{\partial\theta}. \tag{8.123}$$

Setting $f = f_0 + f_1\theta$, where f_0, f_1 are holomorphic functions of z, we see that

$$Df = (-1)^{p(f)}(\frac{\partial f_0}{\partial z}\theta - f_1)$$

and thus

$$K_f = 2f\frac{\partial}{\partial z} + (-1)^{p(f)}(Df)D \tag{8.124}$$

Therefore, K_f lies in $\mathcal{D}$ if and only if $f \equiv 0$. Since D and $\frac{\partial}{\partial z} = \frac{1}{2}D^2$ generate $\mathcal{TM}$ over $\mathcal{O}_\mathcal{M}$, this proves the claim. $\square$

This direct splitting allows us to calculate the Čech cohomology of the sheaf $\mathfrak{k}^L(1|1)$ from that of $\mathcal{D}$, which is easier than a direct attempt.

Proposition 8.4.3. *Let $\mathcal{M}$ be a $\mathfrak{k}^L(1|1)$-surface defined by a Riemann surface M with spin bundle S. Then the first Čech cohomology group of the subsheaf of $\mathfrak{k}^L(1|1)$-vector fields considered as a $\mathbb{Z}_2$-graded $\mathcal{O}_M$-module is given by*

$$H^1(M, \mathfrak{k}^L(1|1)) = H^0(M, K^2) \oplus \Pi(H^0(M, K^{3/2})). \tag{8.125}$$

Proof. From Lemma 8.4.2 we deduce that the cohomology of $\mathcal{TM}$ as a $\mathcal{O}_M$-module of rank $2|2$ can be split as

$$H^q(M, \mathcal{TM}) = H^q(M, \mathcal{D}) \oplus H^q(M, \mathfrak{k}^L(1|1)) \qquad q = 0, 1, \dots. \tag{8.126}$$

Locally, all sections of $\mathcal{D}$ can be written as

$$(f_0 + f_1\theta)(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}) = f_1\theta\frac{\partial}{\partial\theta} + f_0 D. \tag{8.127}$$

The first summand contributes $H^1(M, \mathcal{O}_M) \cong H^0(M, K)$ as the even subspace of $H^1(M, \mathcal{D})$. The second summand contributes $H^1(M, S^{-1}) = H^1(M, K \otimes S) \cong H^0(M, K^{3/2})$ as the odd subspace. Thus, $H^1(M, \mathfrak{k}^L(1|1))$ must be isomorphic to $H^0(M, K^2) \oplus \Pi(H^0(M, K^2 \otimes S^{-1}))$. □

Therefore, the Teichmüller space of $\mathfrak{k}^L(1|1)$-structures ($N = 1$ super Riemann surfaces in physics terminology) has dimension $3g-3|2g-2$, as was already found by many authors, e.g., [Fri86], [Vai84], [CR88], [LR88]. The missing $g|2g-2$ dimensions as compared to the deformations of the supercomplex ($= \mathfrak{vect}^L(1|1)$-)structure alone also have a nice interpretation. The g even dimensions are those of the Jacobian variety, i.e., the vertical dimensions of $J(V_g)$. We cannot deform the structure in these directions, since the divisor class of a spin bundle is rigid (see Prop. 8.2.5). This fact also accounts for the loss of the $2g-2$ odd directions. As one could see in the proofs of Prop. 8.1.3 and Thm. 8.1.4, the subspace $H^0(M, K \otimes L)$ that we lose as valid deformations originates in the component γ_0 of the tangent vector to the complex structure J. The entry γ describes the contribution of $\frac{\partial}{\partial\bar{z}}$ to the infinitesimally deformed vector field $\frac{\partial}{\partial\theta'}$, i.e., to the deformation of the line bundle L. Since the isomorphism class of L must remain constant, no such deformations are possible anymore.

The remaining even deformations describe the ordinary variation of the complex structure of the underlying Riemann surface M, i.e., ordinary Teichmüller space. The odd ones are the true "super"-deformations, which can only occur over a base which contains odd dimensions. They, too, only deform the complex structure of M, but in a way which would be impossible in classical complex geometry, namely by mixing in even combinations of elements of $\Pi(H^0(M, K^{3/2}))$ and the odd variables of the base of the deformation.

Denote by Ξ the set of 2^{2g} disjoint holomorphic sections of $J(V_g) \to \mathcal{T}_g$ corresponding to the divisors of spin bundles defined in Prop. 8.2.5. Each of these sections is biholomorphic to ordinary Teichmüller space $\mathcal{T}_g$, so Ξ can be viewed as a trivial 2^{2g}-sheeted covering of $\mathcal{T}_g$. Then we can restrict the bundle $p : \mathcal{H} \to J(V_g)$ with fiber $H^0(M, K^2 \otimes L^{-1}) \oplus H^0(M, K \otimes L)$ above to the subbundle

$$\begin{aligned} p' : \mathcal{H}' &\to J(V_g) \qquad (8.128)\\ H^0(M, K^2 \otimes L^{-1}) &\to J(V_g). \qquad (8.129)\end{aligned}$$

Then the above discussions make it clear that $\mathcal{T}^g_{\mathfrak{k}^L(1|1)}$ is a subsupermanifold of $\mathcal{T}^{g,g-1}_{\mathfrak{vect}^L(1|1)}$ which contains all isomorphism classes of spin curves and only those odd deformations that preserve the spin bundle.

Theorem 8.4.4. *The supermanifold* $\mathcal{T} = (\Xi, \wedge\mathcal{H}'\big|_{\Xi})$ *parametrizes a semiuniversal family of* $\mathfrak{k}^L(1|1)$*-structures on a smooth closed compact supermanifold of dimension* $2|2$ *whose underlying manifold is a surface of genus* g*.*

Proof. By construction, $\mathcal{T}$ is the closed subsupermanifold of $\mathcal{T}^{g,g-1}_{\mathfrak{vect}^L(1|1)}$ whose underlying manifold parametrizes all isomorphism classes of marked Riemann surfaces of genus g together with a spin bundle, and whose odd directions parametrize all inequivalent odd deformations of such supersurfaces. The semiuniversality of the family parametrized by $\mathcal{T}$ was proven by [Vai88a] and [LR88]. $\square$

In [LR88], the residual $\mathbb{Z}_2$-symmetry (cf. Thm. 8.1.7) is also quotiented out and a "superorbifold" is obtained.

Bibliography

[ABD$^+$76a] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino. Supersymmetric strings and color confinement. *Nucl. Phys. B*, 62(1):105–110, 1976.

[ABD$^+$76b] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R. Pettorino, and J. H. Schwarz. Dual string with $U(1)$ color symmetry. *Nucl. Phys. B*, 111(1):77–110, 1976.

[ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AGV64] M. Artin, A. Grothendieck, and M. Verdier. *Théorie des topos et cohomologie étale des schémas, 1,2,3*, volume 270,305,569 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1963-1964. Séminaire de Géométrie Algébrique du Bois-Marie (SGA4).

[Ati71] M.F. Atiyah. Riemann surfaces and spin structures. *Ann. Sci. École Norm. Sup. (4)*, 4:47–62, 1971.

[Bat79] M. Batchelor. The structure of supermanifolds. *Trans. Amer. Math. Soc.*, 253:329–338, 1979.

[Ber61] L. Bers. Holomorphic differentials as functions of moduli. *Bull. Amer. Math. Soc.*, 67:206–210, 1961.

[Ber73] L. Bers. Fiber spaces over Teichmüller spaces. *Acta. Math.*, 130:89–126, 1973.

[BK01] B. Bakalov and A. Kirillov, Jr. *Lectures on Tensor Categories and Modular Functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[BL75] F. A. Berezin and D. A. Leites. Supermanifolds. *Dokl. Akad. Nauk SSSR*, 224(3):505–508, 1975.

[Cor89] Maurizio Cornalba. Moduli of curves and theta-characteristics. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 560–589. World Sci. Publ., Teaneck, NJ, 1989.

[CR88] L. Crane and J.M. Rabin. Super Riemann Surfaces: Uniformization and Teichmüller Theory. *Comm. Math. Phys.*, 113(4):601–623, 1988.

[Del] P. Deligne. Letter to Yu.I. Manin, dated Sept. 25, 1987. Unpublished.

[DeW92] B. DeWitt. *Supermanifolds*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, second edition, 1992.

[DM99] P. Deligne and J.W. Morgan. Notes on Supersymmetry (following Joseph Bernstein). In *Quantum Fields and Strings: A Course for Mathematicians. Vol 1*, pages 41–97. AMS, Providence, Rhode Island, 1999.

[Dou66] A. Douady. Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):1–95, 1966.

[DP88] E. D'Hoker and D.H. Phong. The geometry of string perturbation theory. *Rev. Mod. Phys.*, 60(4), 1988.

[Ear78] C.J. Earle. Families of Riemann surfaces and Jacobi varieties. *Ann. Math. (2)*, 107(2):255–286, 1978.

[Efe97] K. Efetov. *Supersymmetry in disorder and chaos*. Cambridge University Press, Cambridge, 1997.

[Ehr51] C. Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 29–55. Georges Thone, Liège, 1951.

[FGI$^+$05] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry: Grothendieck's FGA explained*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[Fri86] D. Friedan. Notes on string theory and two-dimensional conformal field theory. In *Workshop on unified string theories (Santa Barbara, Calif., 1985)*, pages 162–213. World Sci. Publishing, Singapore, 1986.

[FS92] H. Flenner and D. Sundararaman. Analytic geometry of complex superspaces. *Trans. Amer. Math. Soc.*, 330(1):1–40, 1992.

[GH94] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[GLS02] P. Grozman, D.A. Leites, and I. Shchepochkina. Invariant operators on supermanifolds and standard models. In *Multiple facets of quantization and supersymmetry*, pages 508–555. World Sci. Publ., River Edge, NJ, 2002.

[GLS05] P. Grozman, D.A. Leites, and I.M. Shchepochkina. Lie Superalgebras of String Theories. *Acta Math. Vietnamica*, 26:27–63, 2005. `arXiv:hep-th/9702120`.

[GM02] S.I. Gelfand and Yu. I. Manin. *Methods of Homological Algebra.* Springer Monographs in Mathematics. Springer Verlag, 2nd edition, 2002.

[GN88] S.B. Giddings and P. Nelson. The geometry of super Riemann surfaces. *Comm. Math. Phys.*, 116(4):607–634, 1988.

[GR84] H. Grauert and R. Remmert. *Coherent Analytic Sheaves*, volume 265 of *Grundlehren.* Springer, Berlin, 1984.

[Gro62] A. Grothendieck. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.].* Secrétariat mathématique, Paris, 1962.

[GSW87] M.B. Green, J.H. Schwarz, and E. Witten. *Superstring theory I, II.* Cambridge University Press, 1987.

[Hod87] L. Hodgkin. A direct calculation of super-Teichmüller space. *Lett. Math. Phys.*, 14(1):47–53, 1987.

[Hod95] L. Hodgkin. The complex structure of supermoduli space. *Classical Quantum Gravity*, 12(9):2135–2148, 1995.

[Hub76] J.H. Hubbard. Sur les sections analytiques de la courbe universelle de Teichmüller. *Mem. Amer. Math. Soc.*, 4(166):ix+137, 1976.

[Huy05] D. Huybrechts. *Complex geometry. An introduction.* Universitext. Springer-Verlag, Berlin, 2005.

[Joh02] P.T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Vol. 1.* Oxford Logic Guides. Clarendon Press, Oxford, 2002.

[Jos06] J. Jost. *Compact Riemann surfaces: An introduction to contemporary mathematics.* Universitext. Springer-Verlag, Berlin, third edition, 2006.

[Kod86] K. Kodaira. *Complex manifolds and deformation of complex structures*, volume 283 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1986.

[KvdL89] V.G. Kac and J.W. van de Leur. On classification of superconformal algebras. In *Strings '88 (College Park, MD, 1988)*, pages 77–106. World Scientific, 1989.

[Le] D.A. Leites (editor). *Super Rieman Surfaces and related structures. With contributions by B. Feigin, T. Khovanova, D.A. Leites, V. Ovsienko, A. Radul, T. Shiota, A. Vaintrob, P. Grozman, C. Sachse, I. Shchepochkina.* Notes of the Jost-Leites seminar SRS, 2004-2006 (to appear).

[Lei80] D.A. Leites. Introduction to the theory of supermanifolds. *Uspekhi Mat. Nauk*, 35(1):1–64, 1980.

[Leiar] D.A. Leites. *Seminar on Supersymmetries, Vol. 1.* Springer Verlag, to appear.

[LR88] C. LeBrun and M. Rothstein. Moduli of super Riemann surfaces. *Comm. Math. Phys.*, 117(1):159–176, 1988.

[LS07] D.A. Leites and C. Sachse. On critical dimensions of string theories. In M. Aslam, H. Faheem, A. Qadir, and H. Saleem, editors, *Proc. of the XII. Regional Conference on Mathematical Physics, Islamabad, Pakistan.* World Scientific, 2007.

[Mac98] S. MacLane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer Verlag, 2nd edition, 1998.

[Man85] Yu. I. Manin. New dimensions in geometry. In *Workshop Bonn 1984 (Bonn, 1984)*, volume 1111 of *Lecture Notes in Math.*, pages 59–101. Springer, Berlin, 1985.

[Man91] Yu.I. Manin. *Topics in noncommutative geometry.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991.

[Man97] Yu. I. Manin. *Gauge field theory and complex geometry*, volume 289 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 1997. Translated from the 1984 Russian original by N. Koblitz and J. R. King, With an appendix by Sergei Merkulov.

[Man04] M. Manetti. Lectures on deformations of complex manifolds (deformations from differential graded viewpoint). *Rend. Mat. Appl. (7)*, 24(1):1–183, 2004.

[Mat92] O. Mathieu. Classification of simple graded Lie algebras of finite growth. *Invent. Math.*, 108(1):455–519, 1992.

[MM92] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer Verlag, 1992.

[Mol] V. Molotkov. Private communications.

[Mol84] V. Molotkov. Infinite-dimensional $\mathbb{Z}_2^k$-supermanifolds. *ICTP preprints*, IC/84/183, 1984. Scanned version available at `http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?198506284`.

[Mol99] V. Molotkov. Glutoses: a Generalization of Topos theory. *arXiv:math.RT/9912060*, 1999.

[Nat04] S. M. Natanzon. *Moduli of Riemann surfaces, real algebraic curves, and their superanalogs*, volume 225 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2004. Translated from the 2003 Russian edition by Sergei Lando.

[NS71] A. Neveu and J. Schwarz. Factorizable dual model of pions. *Nucl. Phys. B*, 31:76–112, 1971.

[Pol05] E. Poletaeva. The analogs of riemann and penrose tensors on supermanifolds. *Preprints of the Max Planck Institute for Mathematics, MPIM 2003-19*, 2005. `arXiv.org:math.RT/0510165`.

[Ram71] P. Ramond. Dual theory for free fermions. *Phys. Rev. D (3)*, 3:2415–2418, 1971.

[Rot85] M.J. Rothstein. Deformations of complex supermanifolds. *Proc. Amer. Math. Soc.*, 95(2):255–260, 1985.

[Sac] C. Sachse. The cocycles defining the Neveu-Schwarz and Ramond superalgebras in terms of superfields (accepted by Theoret.Math.Phys.).

[Sch70] H. Schubert. *Kategorien I.* Springer Verlag, 1970.

[Sch97] T. Schmitt. Supergeometry and quantum field theory, or: what is a classical configuration? *Rev. Math. Phys.*, 9(8):993–1052, 1997.

[Shc99] I.M. Shchepochkina. Five exceptional Lie superalgebras of vector fields. *Funct. Anal. Appl.*, 33(3):209–219, 1999.

[SS87] A. Schwimmer and N. Seiberg. Comments on the $N = 2, 3, 4$ superconformal algebras in two dimensions. *Phys. Lett. B*, 184:191–196, 1987.

[SU02] Y. Shimizu and K. Ueno. *Advances in moduli theory*, volume 206 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2002. Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.

[Tro92] A.J. Tromba. *Teichmüller theory in Riemannian geometry.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992. Lecture notes prepared by Jochen Denzler.

[Vai84] A. Yu. Vaintrob. Deformations of complex structures on supermanifolds. *Funktsional. Anal. i Prilozhen.*, 18(2):59–60, 1984.

[Vai85] A. Yu. Vaintrob. Almost complex structures on supermanifolds. In *Problems in group theory and homological algebra (Russian)*, pages 139–142, 166. Yaroslav. Gos. Univ., Yaroslavl′, 1985. English version in: Seminar on Supermanifolds no. 24, ed. by D.A. Leites, Reports of the Department of Mathematics, University of Stockholm.

[Vai86] A. Yu. Vaintrob. Deformations and moduli of supermanifolds. In *Group-theoretic methods in physics, Vol. 1 (Russian) (Jūrmala, 1985)*, pages 169–174. "Nauka", Moscow, 1986.

[Vai88a] A. Yu. Vaintrob. Deformations of complex structures on supermanifolds. Reports of the Dept. of Mathematics, University of Stockholm, Sweden, Matematiska Institutionen, Stockholms Universitet, Stockholm, Sweden, 1988. Seminar on Supermanifolds No. 24, edited by D.A. Leites.

[Vai88b] A. Yu. Vaintrob. Deformations of complex superspaces and of the coherent sheaves on them. In *Current problems in mathematics. Newest results, Vol. 32*, Itogi Nauki i Tekhniki, pages 125–211. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988. J. Soviet Math. **51** (1990), no. 1, 2140–2188.

[Var04] V.S. Varadarajan. *Supersymmetry for Mathematicians: An Introduction.* AMS, Providence, Rhode Island, 2004.

[Wel73] R. O. Wells, Jr. *Differential analysis on complex manifolds.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1973. Prentice-Hall Series in Modern Analysis.

[Wit82] E. Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.

[WZ74] J. Wess and B. Zumino. Supergauge transformations in four dimensions. *Nuclear Phys.*, B70:39–50, 1974.